Global Focus on Knowledge
Mathematical Science Developing from Figures

Periodicity and Symmetry

Recognizing Symmetry, Recognizing Periodicity

May 19, 2016

Graduate School of Mathematical Sciences

Takashi Tsuboi

東京大学大学院数理科学研究科
Graduate School of Mathematical Sciences, THE UNIVERSITY OF TOKYO

東京大学
THE UNIVERSITY OF TOKYO

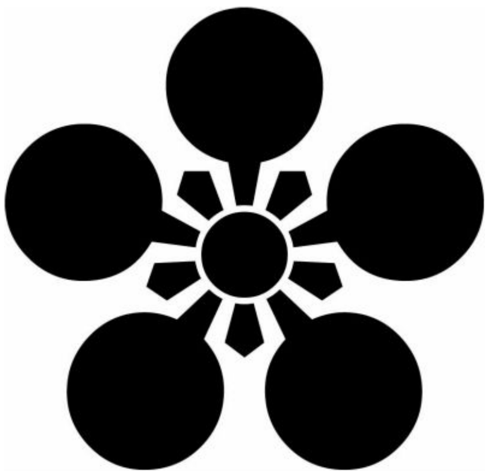
Sekigahara Kassen Byōbu, Collection of The City of Gifu Museum of History（from Wikimedia Commons）

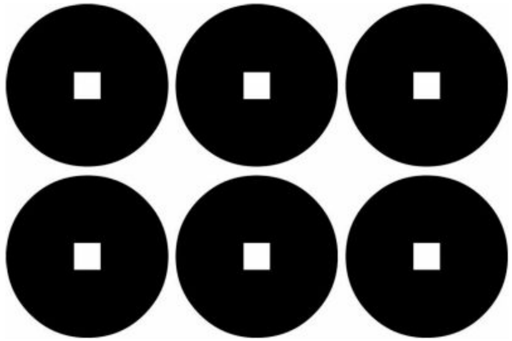
東京大学大学院数理科学研究科
Graduate School of Mathematical Sciences, THE UNIVERSITY OF TOKYO

東京大学
THE UNIVERSITY OF TOKYO

In battles in old days, warriors showed their families by flags with family crests

We can distinguish their identities by the figures





東京大学大学院数理科学研究科
Graduate School of Mathematical Sciences, THE UNIVERSITY OF TOKYO

東京大学
THE UNIVERSITY OF TOKYO

東京大学 大学院総合文化研究科・教養学部

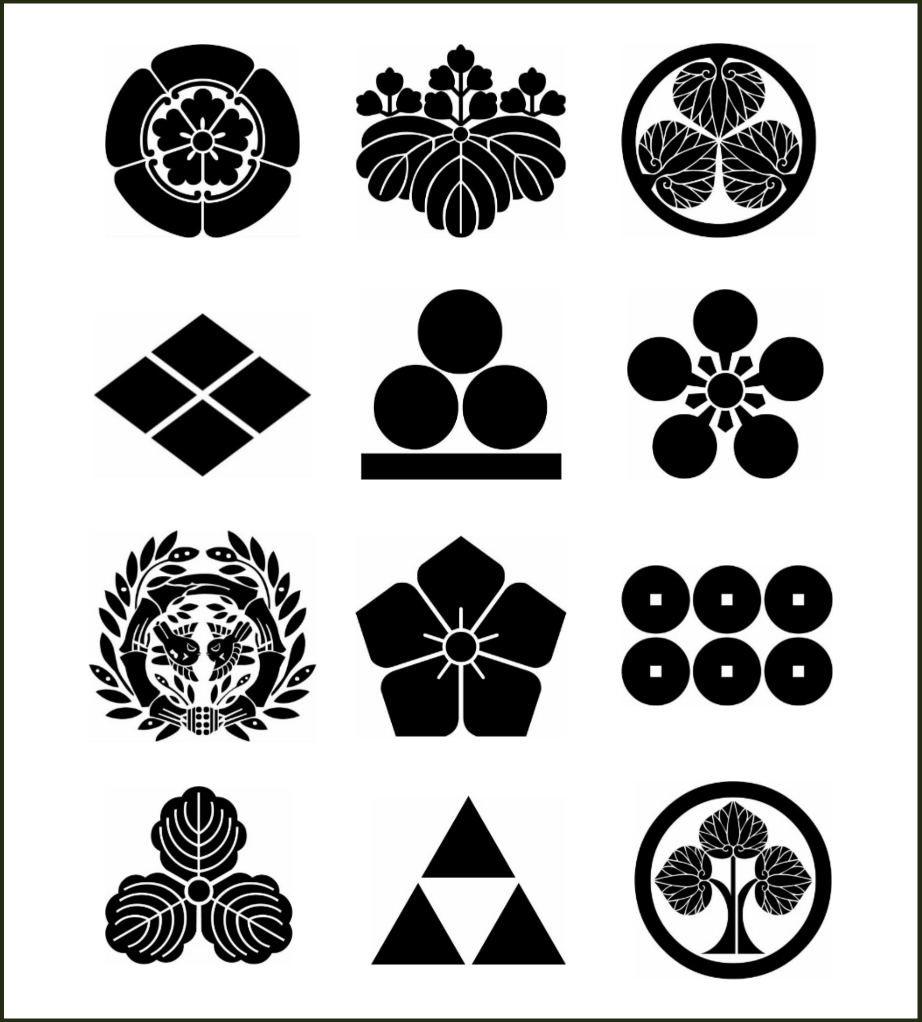
The University of Tokyo, Komaba

Graduate School of Arts and Sciences, College of Arts and Sciences

Courtesy of the University of Tokyo, Graduate School of Arts and Sciences, College of Arts, and Sciences

The oak leaves in the center is the symbol of Daiichi Kôtôgakkou which moved to Komaba in 1935.

From http://museum.c.u-tokyo.ac.jp/ICHIKOH/history04.html, Courtesy of Komaba Museum

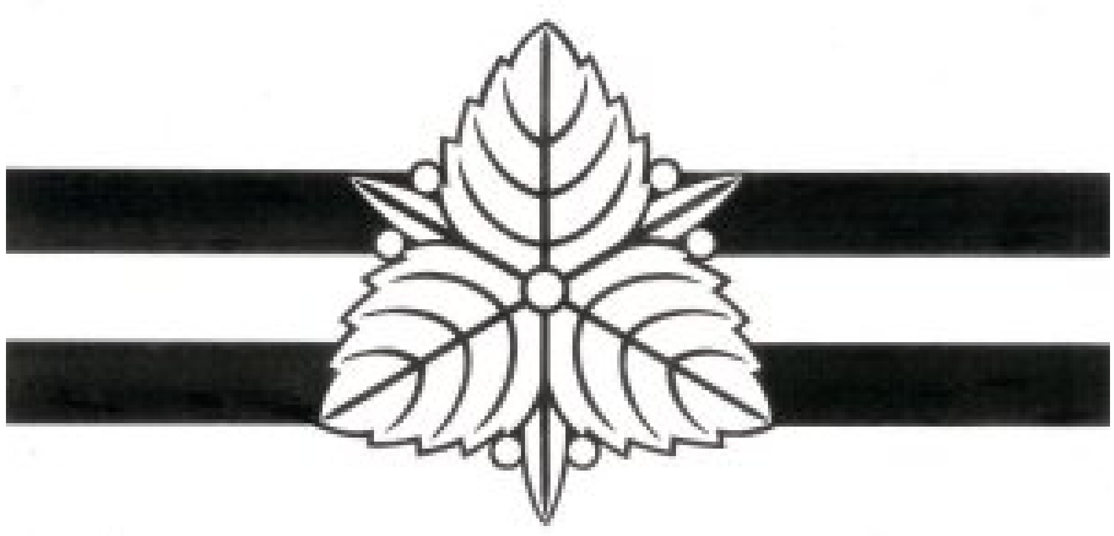


Is there some reason that family crests usually have symmetry ?

In the sense of sight, the human brain seems to be programmed to recognize symmetry.

1. What is symmetry ?

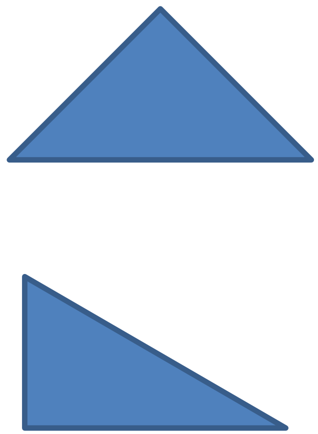
Line symmetry: Point symmetry: Rotational symmetry

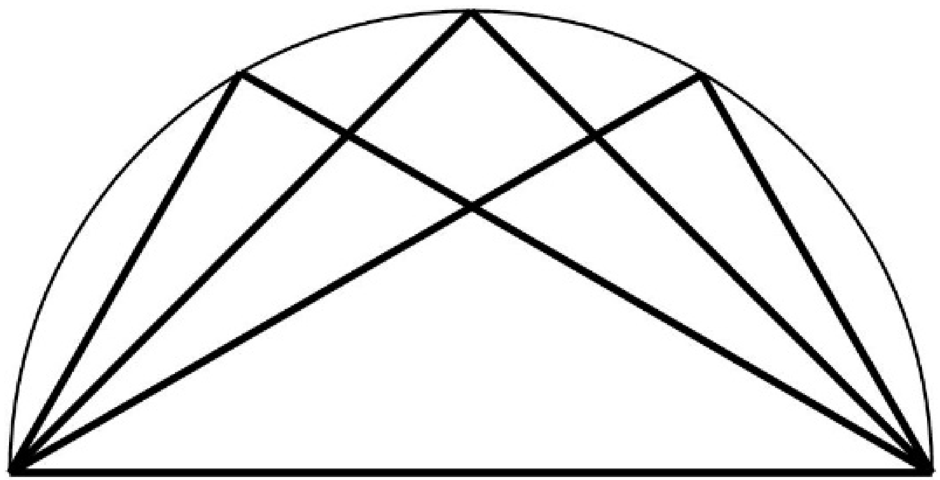


From http://museum.c.u-tokyo.ac.jp/ICHIKOH/history04.html, Courtesy of Komaba Museum

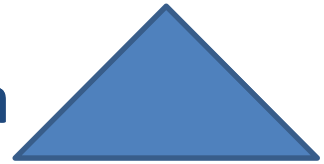
In the explanation of regular polyhedra in Lecture 1 by Prof Kanai appeared the figure ‘Stoicheia’.

These are the figures learned first in elementary school.

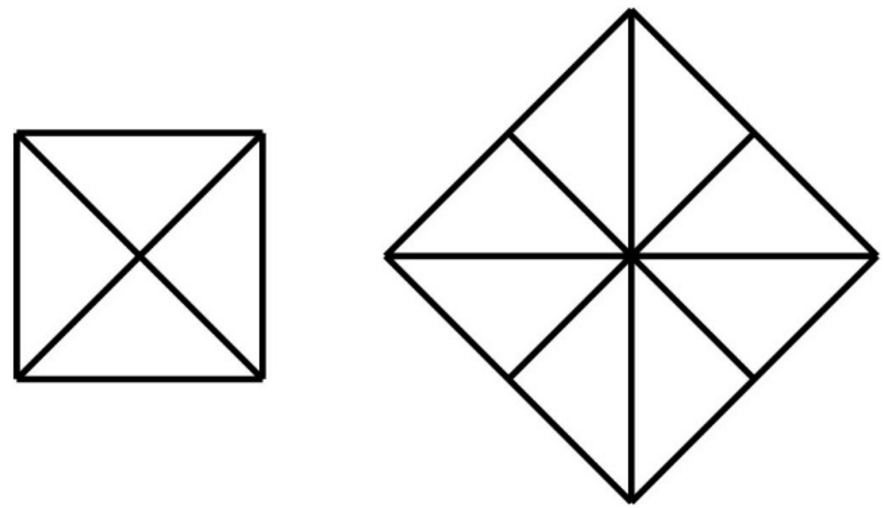




‘Stoicheia’ means ‘elements’, ‘atoms’, …

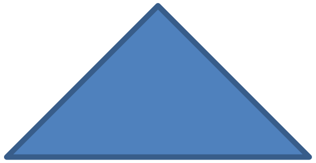
Question: Make figures from many copies of the Stoicheia  of the same size by attaching along edges around a point (the origin O). How many such figures exist ?

Answer: 2.



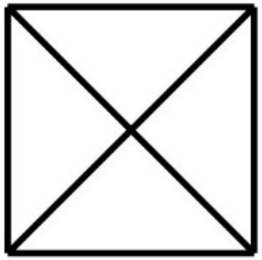
东京大学大学院数理科学研究科
Graduate School of Mathematical Sciences, THE UNIVERSITY OF TOKYO

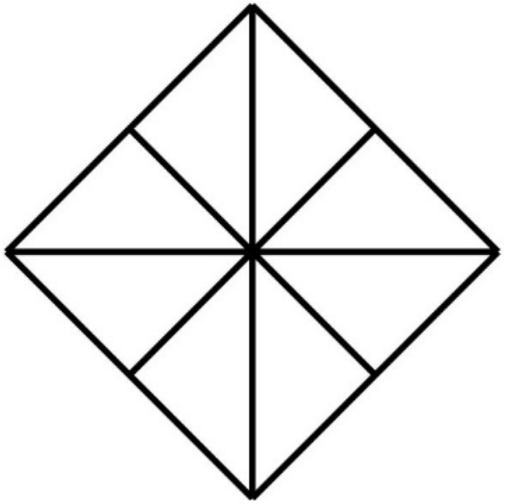
東京大学
THE UNIVERSITY OF TOKYO

Question: Make figures from many copies of the Stoicheia  of the same size by attaching along edges around a point (the origin O). How many such figures exist ?

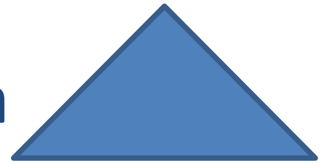
Answer:

Answer: 6.



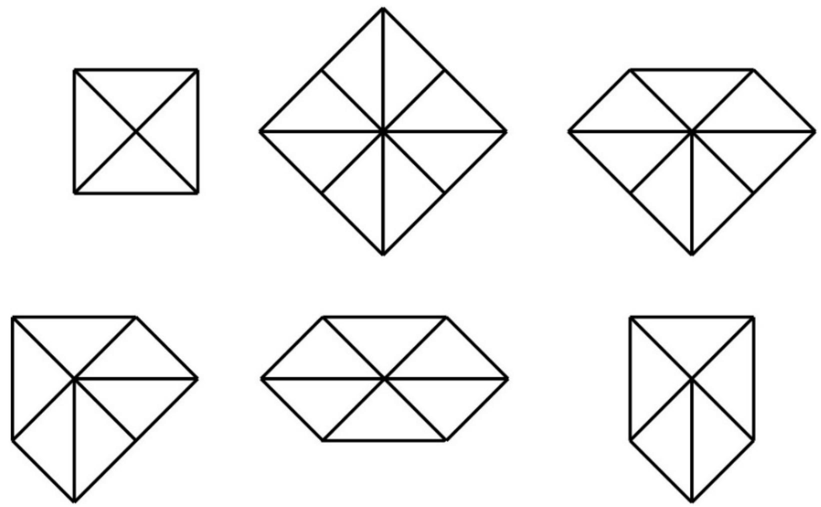


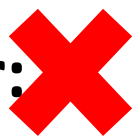
Question: Make figures from many copies of the

Stoicheia  of the same size by

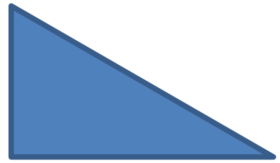
attaching along edges around a point

(the origin O). How many such figures exist ?



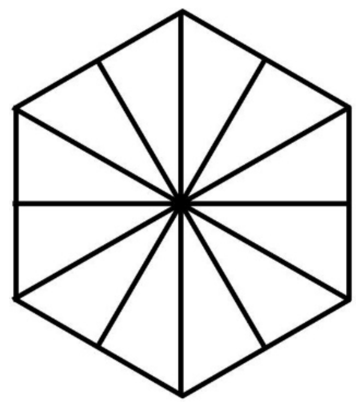
Answer: 

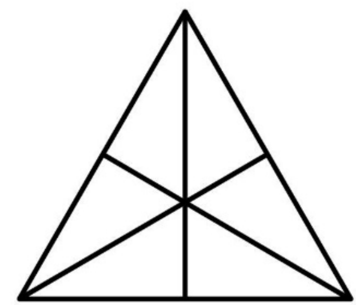
Answer: 6.

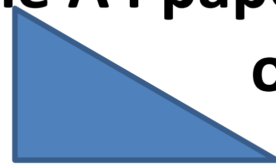
Question: Make figures from many copies of the Stoicheia  of the same size by attaching along edges around a point (the origin O). How many such figures exist ?

Answer: ???????

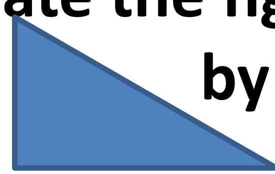
There are two regular polygons, there is a diamond, and more …



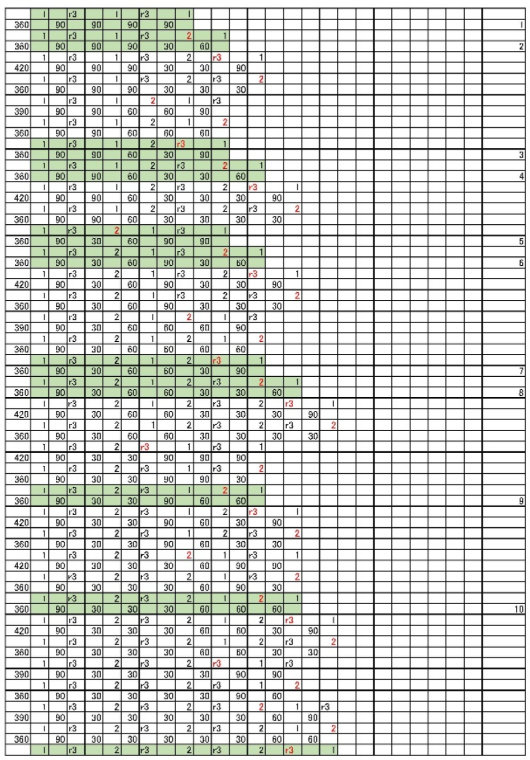


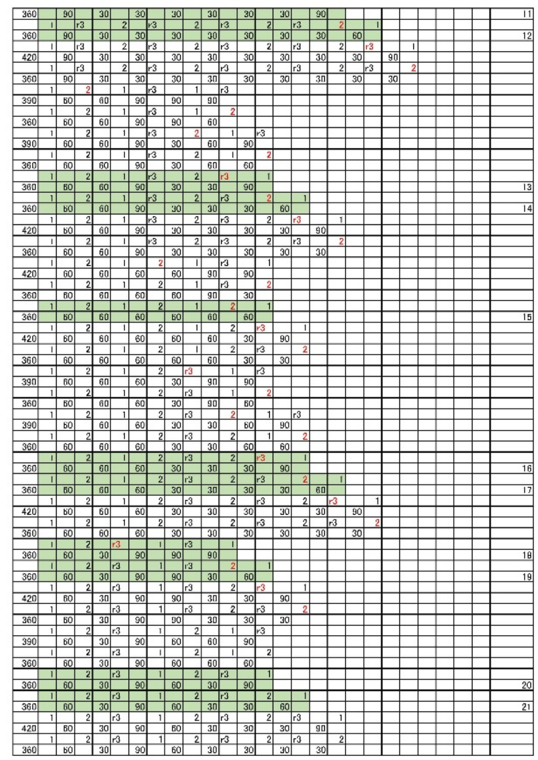
Exercise: Fold a A4 paper to make a regular triangle.
Then cut the A4 paper by scissors and make 12 Stoicheia  of the same size.

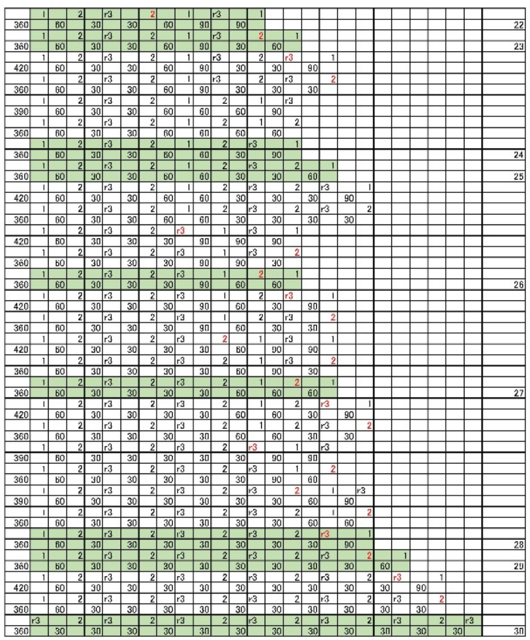
Exercise: By attaching them along the edges around a point and make figures without symmetries (not line symmetric nor point symmetric nor rotationally symmetric).

Exercise: Enumerate the figures made from the Stoicheia  by attaching along edges around a point ?

Enumeration by using an ordering

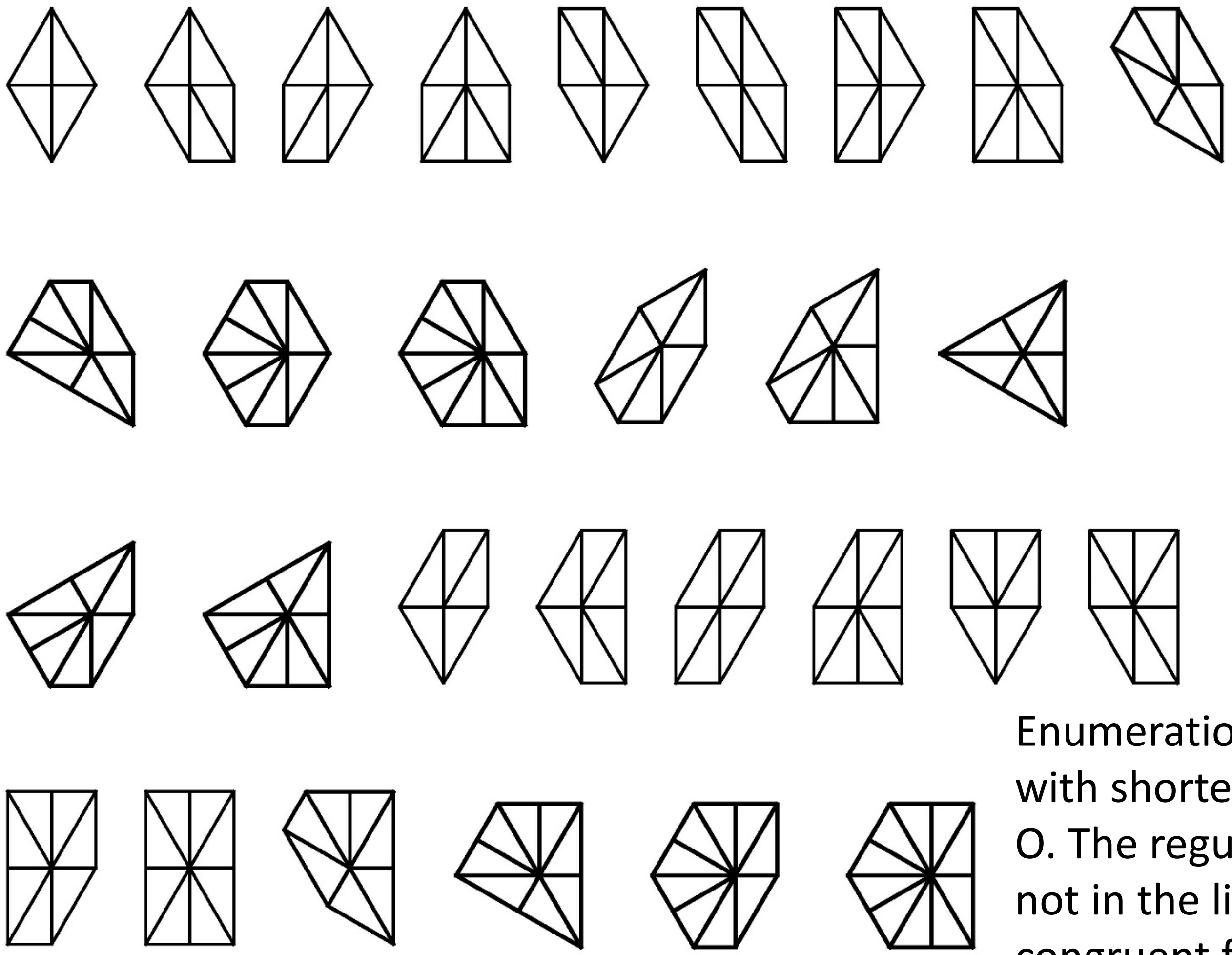




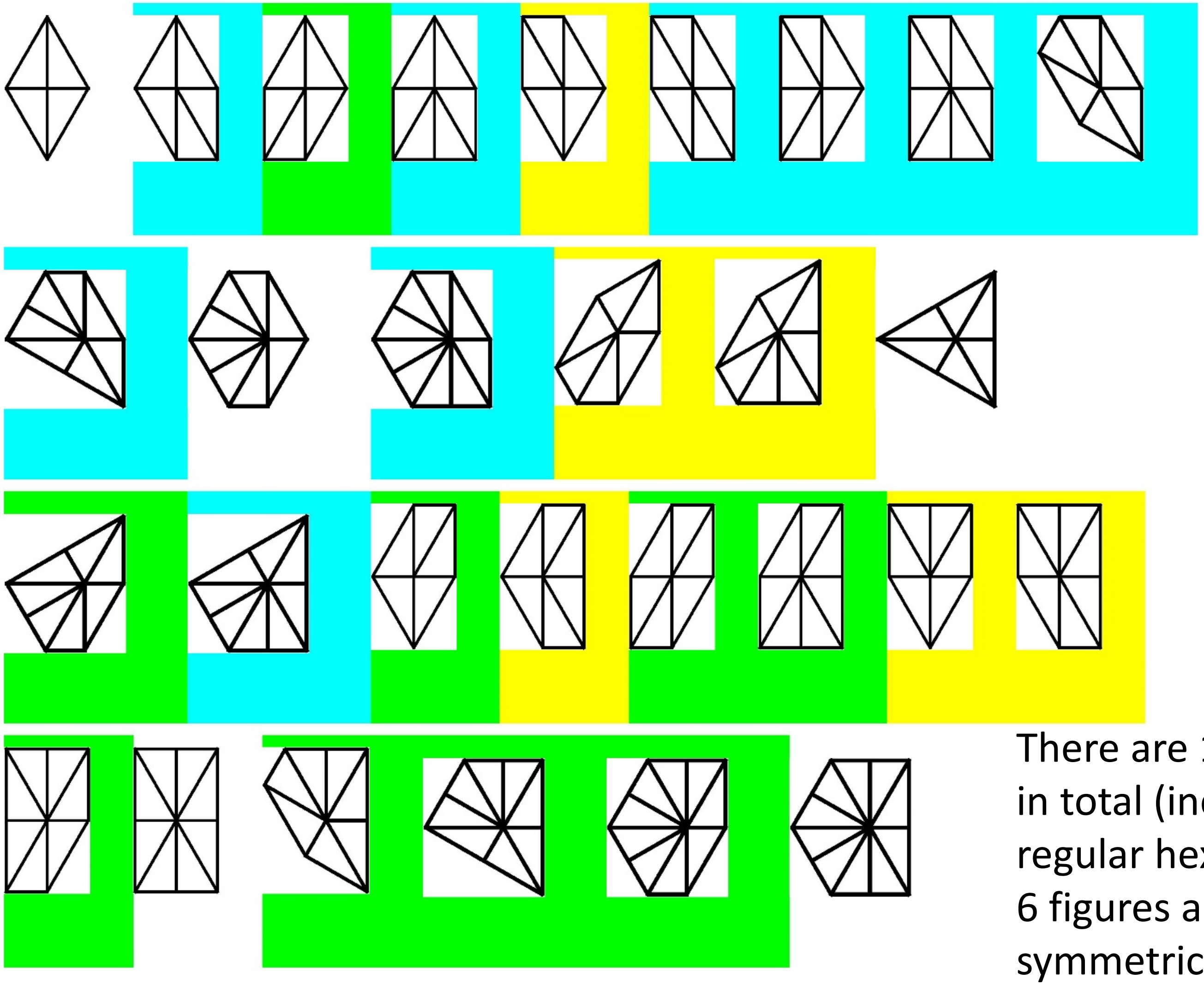


東京大学大学院数理科学研究科
Graduate School of Mathematical Sciences, THE UNIVERSITY OF TOKYO

東京大学
THE UNIVERSITY OF TOKYO



Enumeration of figures with shortest edge from O. The regular hexagon is not in the list. There are congruent figures by rotations or reflections.



There are 15 figures in total (including regular hexagon).
6 figures are not symmetric.

As for the symmetries of planar figures, why they are point symmetric, line symmetric or rotationally symmetric ?

To consider this question, we need terminology to describe symmetry.

Definition: Two figures F , F' are congruent if one can be moved exactly onto the other.

This definition relies on the word 'move' to define the congruence.

東京大学大学院数理科学研究科
Graduate School of Mathematical Sciences, THE UNIVERSITY OF TOKYO
東京大学
THE UNIVERSITY OF TOKYO

2. Moves in the Euclidean space

Definition. A "move" of a figure F in the Euclidean space R^3 is a map $f : F \longrightarrow R^3$ which preserves the distance of 2 points (the length of the line segment joining 2 points). $(\|f(\vec{x}) - f(\vec{y})\| = \|\vec{x} - \vec{y}\|)$

Here $\|\vec{v}\|$ denotes the norm of the vector, that is, for

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \ \|\vec{v}\| = \sqrt{{}^t\vec{v}\,\vec{v}} = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}.$$

Proposition. A move $f : F \longrightarrow R^3$ of a figure F in the Euclidean space can be extended to a move of the Euclidean space $\widehat{f} : R^3 \longrightarrow R^3$.

東京大学大学院数理科学研究科 Graduate School of Mathematical Sciences, THE UNIVERSITY OF TOKYO

東京大学 THE UNIVERSITY OF TOKYO

Proposition. Let $\vec{b} \in R^3$ and let A be a 3×3 matrix such that ${}^tA\,A = I$ (an orthogonal matrix).
Then $\vec{x} \longmapsto A\vec{x} + \vec{b}$ preserves the length of a line segment.

If it is not in dimension 3 but in dimension 2, A is written as $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and

$${}^tA\,A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2+c^2 & ab+cd \\ ab+cd & b^2+d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus either $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, or $A = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$.

Proposition. Let $\vec{b} \in R^3$ and let A be a 3×3 matrix such that ${}^tA\,A = I$ (an orthogonal matrix).
Then $\vec{x} \longmapsto A\vec{x} + \vec{b}$ preserves the length of a line segment.

The proof of the proposition is easy.

For $\vec{v} \in R^3$, $\|A\vec{v}\| = \|\vec{v}\|$. For,

$$\|A\vec{v}\|^2 = {}^t(A\vec{v})A\vec{v} = {}^t\vec{v}\;{}^tAA\vec{v} = {}^t\vec{v}\;I\vec{v} = {}^t\vec{v}\;\vec{v} = \|\vec{v}\|^2.$$

Length preserving is shown as follows:

$$\begin{aligned}\|(A\vec{x} + \vec{b}) - (A\vec{y} + \vec{b})\| &= \|A\vec{x} - A\vec{y}\| \\ &= \|A(\vec{x} - \vec{y})\| = \|\vec{x} - \vec{y}\|.\end{aligned}$$

東京大学大学院数理科学研究科 Graduate School of Mathematical Sciences, THE UNIVERSITY OF TOKYO

東京大学 THE UNIVERSITY OF TOKYO

Thus we have shown that for an orthogonal matrix A, $\vec{x} \longmapsto A\vec{x} + \vec{b}$ is a move of the Euclidean space.

Conversely, any move of the Euclidean space can be written in this form.

Let us carry out an experiment for the Euclidean plane.

For the Euclidean plane, any orientation preserving move is either a parallel translation or a rotation around a point.

Please make a small move of the transparency on the paper.

Can you see some pattern ?

Proposition. Any move $f : R^3 \longrightarrow R^3$ of the Euclidean space can be written as the composition of a rotation or a reflection and a parallel translation: $f(\vec{x}) = A\vec{x} + \vec{b}$.

The proposition in the 2-dimensional case implies that any orientation preserving move is either a parallel translation or a rotation around a point. If the move reverses the orientation, it is a composition of a reflection with respect to a line and a translation in the direction of the line.

This answers the question "why they are point symmetric, line symmetric or rotationally symmetric ? "

A move in the 3-dimensional Euclidean space is one of the followings.

- a parallel translation
- the composition of a rotation around a line and a translation in the direction of the line.
- the composition of a reflection with respect to a plane and a rotation around a line perpendicular to the plane or a translation in a direction in the plane.

Proof. For a move $f : R^3 \longrightarrow R^3$, put $f(\vec{0}) = \vec{b}$. Then $\vec{x} \longmapsto f(\vec{x}) - \vec{b}$ preserves the length, and hence is a move such that $\vec{0} \longmapsto \vec{0}$. We replace f by this move.

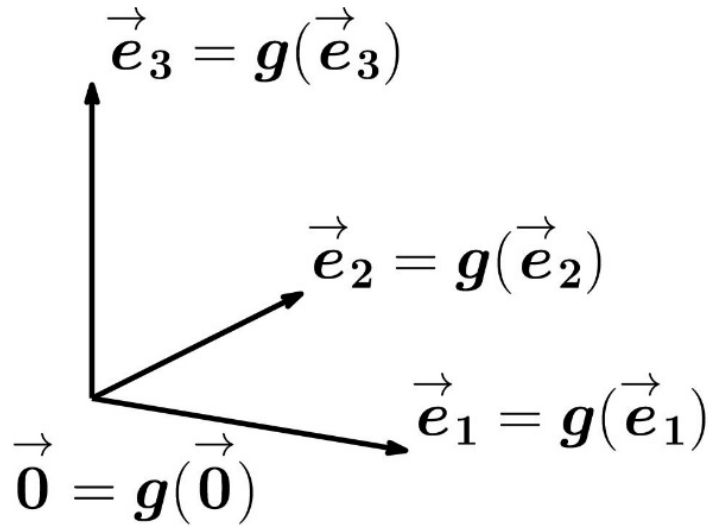
For the basis vectors $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$, $f(\vec{e}_1)$, $f(\vec{e}_2)$ and $f(\vec{e}_3)$ are vectors of norm 1 orthogonal to each other. We write them as column vectors and the matrix $A = \begin{pmatrix} f(\vec{e}_1) & f(\vec{e}_2) & f(\vec{e}_3) \end{pmatrix}$ satisfies ${}^tA\ A = I$.

Let $g : R^3 \longrightarrow R^3$ be the map defined by $g(\vec{x}) = {}^tA\ f(\vec{x})$. Then g maps the origin and the basis vectors to themselves. That is, $g(\vec{0}) = \vec{0}$, $g(\vec{e}_1) = \vec{e}_1$, $g(\vec{e}_2) = \vec{e}_2$ and $g(\vec{e}_3) = \vec{e}_3$.

東京大学大学院数理科学研究科
Graduate School of Mathematical Sciences, THE UNIVERSITY OF TOKYO
東京大学
THE UNIVERSITY OF TOKYO

For any vector $\vec{x}$, the distances between $\vec{x}$ and $\vec{0}$, $\vec{e}_1$, $\vec{e}_2$ or $\vec{e}_3$ are preserved by g. That is, $g(\vec{x})$ satisfies
$\|g(\vec{x})\| = \|g(\vec{x}) - g(\vec{0})\| = \|\vec{x} - \vec{0}\| = \|\vec{x}\|$, and
$\|g(\vec{x}) - \vec{e}_i\| = \|g(\vec{x}) - g(\vec{e}_i)\| = \|\vec{x} - \vec{e}_i\|$ $(i = 1,\ 2,\ 3)$.
Then $g(\vec{x}) = \vec{x}$, that is, g is the identity map.

This implies that $f(\vec{x}) = ({}^tA)^{-1}\vec{x} = A\vec{x}$ and the original f is written as $f(\vec{x}) = A\vec{x} + \vec{b}$.



东京大学大学院数理科学研究科 Graduate School of Mathematical Sciences, THE UNIVERSITY OF TOKYO 東京大学 THE UNIVERSITY OF TOKYO

The last assertion is the principle of survey in geography.

Take your time to think about the way to show it.

This principle is valid not only in the Euclidean space but in the space of constant negative curvature (the hyperbolic space) as well as in the space of constant positive curvature (the sphere).

It can be shown by solving the algebraic equation and it can also be shown geometrically by an induction on the dimension using the facts that two points determine a line, three points determine a plane, the intersection of two spheres is a circle, ...

3. Representation of the symmetry by the isometry group

Consider the set I(F) of moves of a figure F to F itself.

(A 'move' is also called an 'isometry' or an 'isometric transformation'.)

The identity map id_F of F belongs to $I(F)$, and the inverse map f^{-1} of an element f of $I(F)$ also belongs to $I(F)$.
For elements f and g of $I(F)$, their composition $f \circ g$ belongs to $I(F)$.

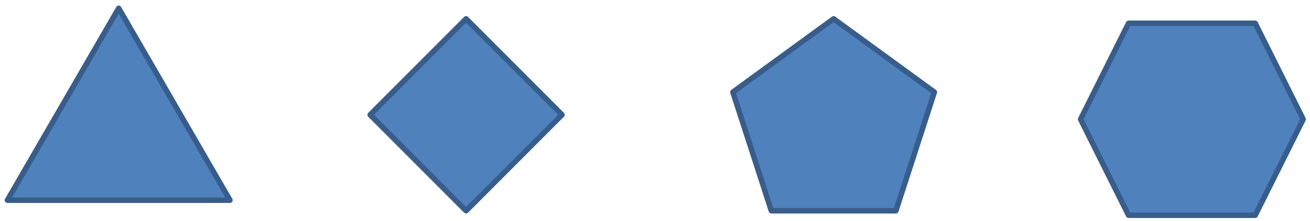
For the composition $\circ$, the equalities $f\circ(g\circ h) = (f\circ g)\circ h$ (associative law), $\mathrm{id}_F \circ f = f = f \circ \mathrm{id}_F$ and $f^{-1} \circ f = \mathrm{id}_F = f \circ f^{-1}$ hold.

Hence $I(F)$ is a group where the identity element is id_F and the inverse element of f is f^{-1}.

(In general, the set G with a map $G \times G \longrightarrow G$ which satisfies the associative law is called a group if it has the identity element and the inverse element of each element.)

We call $I(F)$ the isometry group or the congruence group of the figure F.

Proposition. The isometry group of a bounded planar figure is isomorphic either to C_k or to D_{2k}.



C_3 , D_6 C_4 , D_8 C_5 , D_{10} C_6 , D_{12}

D_{2k} is the isometry group of the regular k-gon and C_k is the subset of D_{2k} consisting of those elements which preserve the orientation (that is, those which preserve the orientation of the edges which is given counterclockwise).

We show that the isometry group of a bounded planar figure is isomorphic to a subgroup of the orthogonal group.

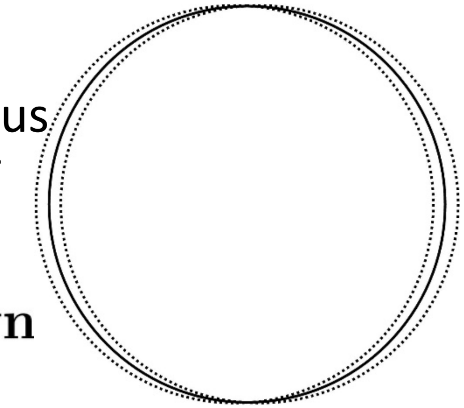
- F is bounded if it is contained in a disk of finite radius with the center being the origin.
- Consider the closed disk of the minimum radius which contains the figure F.
- Such a closed disk D of the minimum radius uniquely exists.
- By the existence and the uniqueness of the closed disk D of the minimum radius, $I(F) \subset I(D)$. Here $\subset$ is the symbol for the subset which has the same meaning as $\subseteq$.

東京大学大学院数理科学研究科
Graduate School of Mathematical Sciences, THE UNIVERSITY OF TOKYO

東京大学
THE UNIVERSITY OF TOKYO

To show that the uniqueness and the existence of the closed disk D of the minimum radius, we use properties of the real numbers.

The uniqueness of the closed disk D of the minimum radius follows from the fact that the intersection of two disks of the same radius is contained in a disk of a smaller radius.



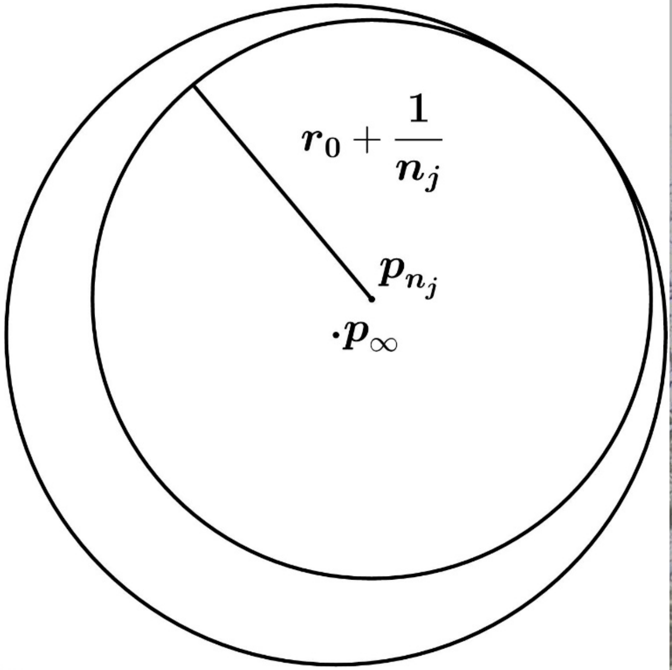
The existence of the minimum radius is shown as follows:

- Put $A = \{r \mid$ there exists a closed disk of radius r containing the figure $F\}$. Then A is a set of positive real numbers, and hence the infimum $r_0 = \inf A$ is defined.

  The infimum r_0 is the largest real number such that $r_0 \leq r$ holds for any element r of A.

- It is necessary to show that there exists a closed disk of radius r_0 which contains F.
- Since r_0 is the infimum, for any positive real number ε, there exists a closed disk of radius (not greater than) $r_0 + \varepsilon$ which contains F.
- Hence, for any positive integer n, there exists a closed disk centered at p_n of radius $r_0 + \dfrac{1}{n}$ which contains F.
- Since $\{p_n\}$ is a bounded sequence of points, there exists a subsequence $\{p_{n_j}\}$ which converges. Let p_∞ denote the limit of this subsequence: $\lim_{j\to\infty} p_{n_j} = p_\infty$.

- Then the disk centered at p_∞ of radius $r_0 + \dfrac{1}{n_j} + \|p_{n_j} - p_\infty\|$ contains F.



- As $j \to \infty$, $r_0 + \dfrac{1}{n_j} + \|p_{n_j} - p_\infty\| \to r_0$. Hence F is contained in the intersection of these disks, which is the disk centered at p_∞ of radius r_0.

In order to show

__Proposition.__ The isometry group of a bounded planar figure is isomorphic either to C_k or to D_{2k}.

we have shown

__Proposition.__ The isometry group of a bounded planar figure is isomorphic to a subgroup of the isometry group of a disk.

Now it is necessary to show

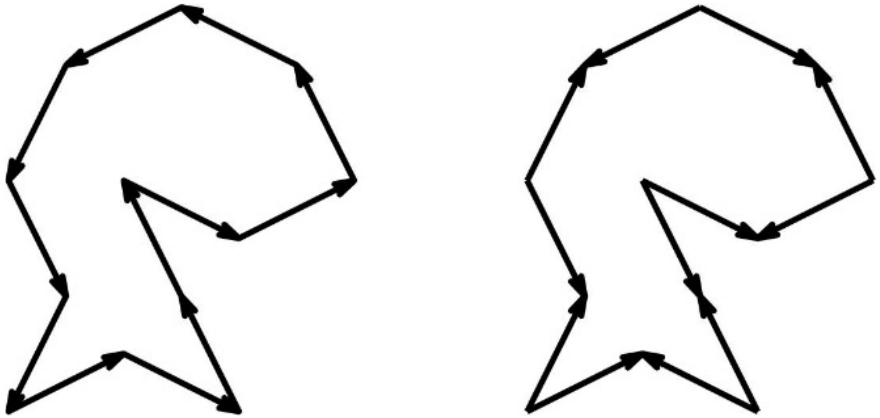
__Proposition.__ Any finite subgroup of the isometry group of a disk is isomorphic either to C_k or to D_{2k}.

When the group contains rotations, this can be shown by looking at the minimum of the positive rotation angles of elements of the group.

For further study, we explain the "orbit space" of a group action, which is not very elementary.

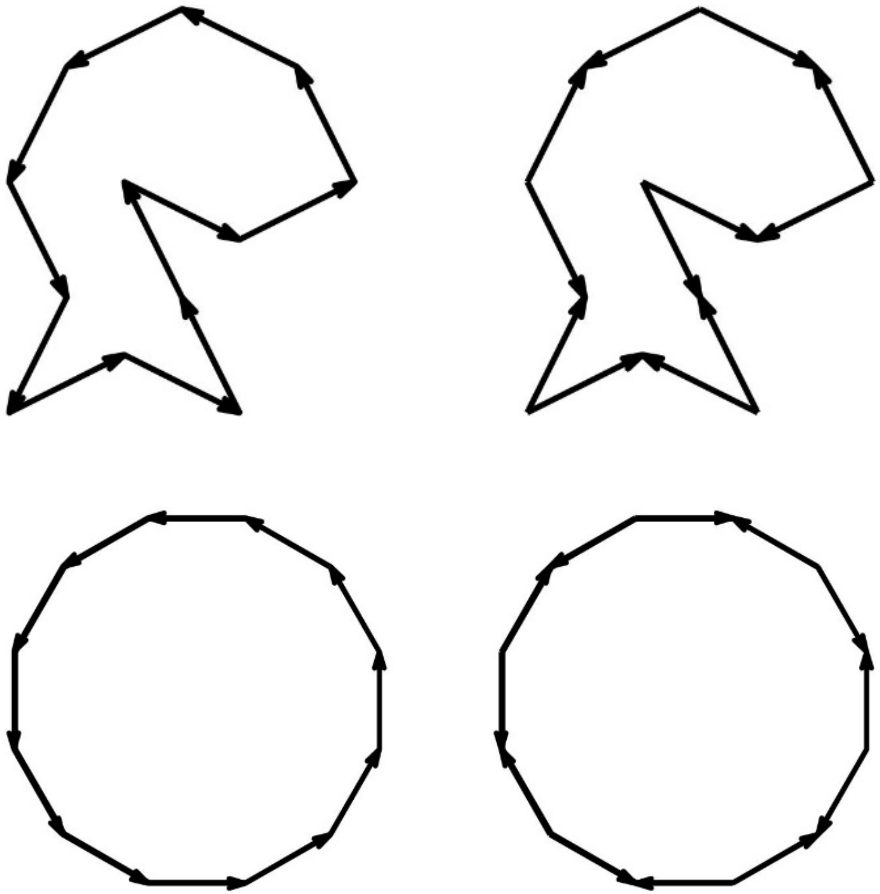
The groups C_k and D_{2k} act on the 1-dimensional figures consisting of k arrows (oriented line segments) and $2k$ arrows attached as in the figures sending arrows to arrows, respectively.

In the figures, C_{12} and D_{12} act on the closed piecewise linear curves consisting of 12 oriented line segments, respectively.



Fix an arrow and pick an arrow to move to, then this determines an element of C_{12} or D_{12}.

The orbit of a point consists of the points whose positions on arrows are the same. Hence the orbit space is a circle or a line segment.



Assume that a group G acts isometrically on the circle S^1.

When the isometry $f \in G$ is a rotation, f has no fixed points on the circle S^1, a small arc containing a point $x \in S^1$ is sent to a small arc containing $f(x)$ and these arcs are disjoint.

When the isometry f is a reflection with respect to a line passing through the origin, there are two fixed points. If x is not a fixed point, there is a small arc containing x sent to a disjoint small arc containing $f(x)$. A small arc containing a fixed point is folded.

Let us think about the set of elements of the circle S^1 sent to others by elements of G.

The set of points sent to others by elements of G is called the orbit, and the space consisting of orbits is called the orbit space.

This means that points x and y are identified if there is an element $f \in G$ such that $y = f(x)$.

If a point $x \in S^1$ is a fixed point of a reflection belonging to G, it corresponds to an end point of the orbit space.

If a point $x \in S^1$ is not a fixed point of isometries of G other than id_{S^1}, it is an interior point of the orbit space.

As a conclusion, the orbit space of the action of a finite isometry group G on the circle is either a circle or a closed interval.

Let a group G act on the real line $\boldsymbol{R}$ so that the orbit space $\boldsymbol{R}/G$ is 1-dimensional and compact. Then there are the following two cases.

- G is a group generated by a translation T and is isomorphic to $C_\infty \cong \boldsymbol{Z}$.

$$\cdots \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdots$$

  The orbit space $\boldsymbol{R}/C_\infty$ is homeomorphic to a circle.

- G is a group generated by two reflections r_0 and r_1 and is isomorphic to $C_2 * C_2 \cong \boldsymbol{Z} \rtimes \boldsymbol{Z}_2$.

$$\cdots \longrightarrow \cdot \longleftarrow \cdot \longrightarrow \cdot \longleftarrow \cdot \longrightarrow \cdot \longleftarrow \cdots$$

  The orbit space $\boldsymbol{R}/(C_2 * C_2)$ is homeomorphic to a closed interval.