学術俯瞰講義

主題科目／学術フロンティア講義　2単位　1、2年生対象

図形から拡がる数理科学

コーディネータ：坪井俊（理学部）
ナビゲータ：金井雅彦（理学部）

坪井俊　金井雅彦　舘知宏　齋藤宣一　五十嵐健夫　河野俊丈

多面体をめぐって　金井雅彦（理学部）

第1回	4/7	オイラーの公式と正多面体
第2回	4/14	コーシーの剛性定理と変形する多面体
第3回	4/21	多面体に関わる未解決問題

折紙の数理と計算　舘知宏（教養学部）

第4回	4/28	計算折紙とかたち
第5回	5/12	折紙のメカニズム

周期性と対称性　坪井俊（理学部）

第6回	5/19	対称性を見つける・周期性を見つける
第7回	5/26	回転と平行移動で空間を理解する
第8回	6/9	空間を埋め尽くす多面体

シミュレーションと多角形・多面体　齋藤宣一（理学部）

第9回	6/16	シミュレーションと多角形・多面体

対話的形状モデリング　五十嵐健夫（理学部）

第10回	6/23	対話的形状モデリング

4次元多面体から空間のかたちをみる　河野俊丈（理学部）

第11回	6/30	4次元の図形を見よう
第12回	7/7	空間が曲がっているとはどういうことか
第13回	7/14	宇宙のトポロジー

駒場キャンパス　21 KOMCEE レクチャーホール　木曜日　5時限（16:50-18:35）

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Global Forcus on Knowledge
Mathematical Science Developing from Figures

Periodicity and Symmetry

Understanding the space by rotations and translations

May 26, 2016

Graduate School of Mathematical Sciences
Takashi Tsuboi

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東京大学
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We studied the figures on the real line with periodicity and symmetry.

Let a group G act on the real line R so that the orbit space R/G is 1-dimensional and compact. Then there are the following two cases.

- G is a group generated by a translation T and is isomorphic to $C_\infty \cong Z$.

$$\cdots \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdot \longrightarrow \cdots$$

The orbit space R/C_∞ is homeomorphic to a circle.

- G is a group generated by two reflections r_0 and r_1 and is isomorphic to $C_2 * C_2 \cong Z \rtimes Z_2$.

$$\cdots \longrightarrow \cdot \longleftarrow \cdot \longrightarrow \cdot \longleftarrow \cdot \longrightarrow \cdot \longleftarrow \cdots$$

The orbit space $R/(C_2 * C_2)$ is homeomorphic to a closed interval.

1. How are two dimensional figures with periodicity and symmetry ?

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Maurits Cornelis Escher（Devil）

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Maurits Cornelis Escher（Butterflies）

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Maurits Cornelis Escher（Horseman）

1. How are two dimensional figures with periodicity and symmetry ?

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Maurits Cornelis Escher
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Maurits Cornelis Escher
（Fish and Devil）

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Maurits Cornelis Escher
（Crab）

1. How are two dimensional figures with periodicity and symmetry ?

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Maurits Cornelis Escher
（Fish-Watercolor）

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Maurits Cornelis Escher
（Birds）

Let us understand the designs by Esher (1896-1972) as repetitions of designs on a smallest polygons (as simple as possible).

Please find the smallest unit of the designs distributed by using rulers.

First find a parallelogram which represents the periodicity and examine it whether it can be divided into smaller pieces.

Let us not distinguish the colors of butterflies.

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Maurits Cornelis Escher（Devil）

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Maurits Cornelis Escher（Butterflies）

Let us not distinguish colors

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Maurits Cornelis Escher
（Starfish and Clams）

Let us not distinguish colors of crabs

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Maurits Cornelis Escher
（Fish and Devil)

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Maurits Cornelis Escher
（Crab）

Let us not distinguish colors of fish

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Maurits Cornelis Escher
（Fish）

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Maurits Cornelis Escher
（Fish-Watercolor）

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Maurits Cornelis Escher
（Birds）

When we look for the unit pattern of the figures with periodicity and symmetry, we often find “stoicheia”.

It is easy to see that we obtain figures with periodicity and symmetry from copies of “stoicheia” by attaching along their edges.

It is only a sufficient condition, and this does not explain the necessity of the appearance of “stoicheia”.

Definition of the periodicity and lattices

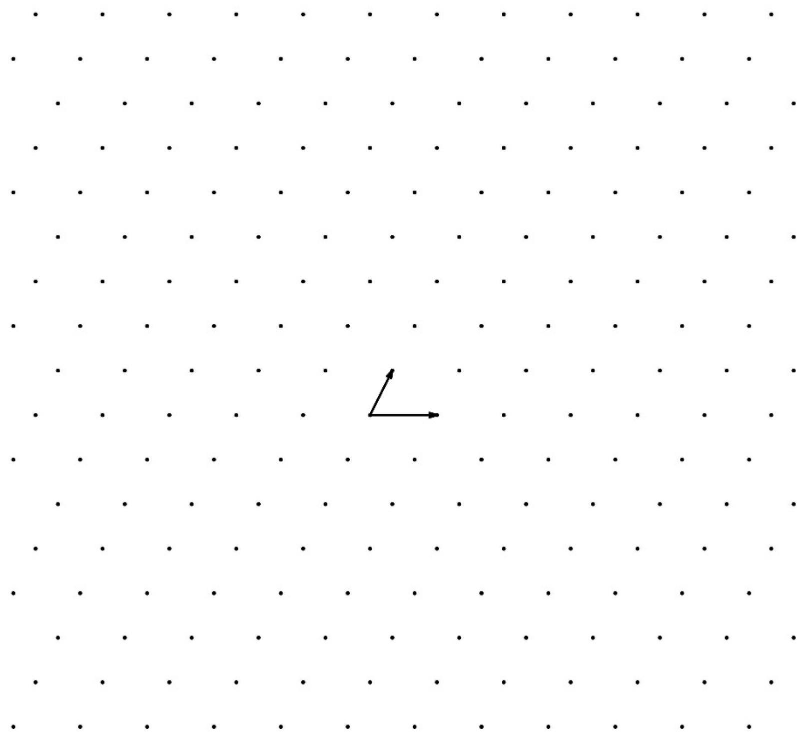
Definition of the figures with periodicity. A figure on the plane with periodicity is a figure such that its isometry group (congruence group) contains two independent translations.

Problem. Classify the isometry groups (congruence groups) of the figures on the plane with periodicity and symmetry.

It is know that there are 17 types of the groups of isometry of such figures.

2. Lattices

For two linearly independent vectors $\vec{a}$ and $\vec{b}$, the set $L = \{m\vec{a} + n\vec{b} \mid m, n \in Z\}$ of the sums of their integer multiples is called a lattice on the plane.



The parallelotoid with vertices $\vec{0}$, $\vec{a}$, $\vec{b}$, $\vec{a}+\vec{b}$ is determined up to similarity transformations by the ratio $\dfrac{\|\vec{a}\|}{\|\vec{b}\|}$ of the norms of the two vectors and the angle formed by the vectors. When the ratio is replaced by the inverse (reciprocal) number or the angle θ is replaced by the supplementary angle $\pi-\theta$, the paralleloids are similar.

The lattice is formed by the vertices when the plane is tiled by paralleloids. Then different paralleloids determine the same lattice.

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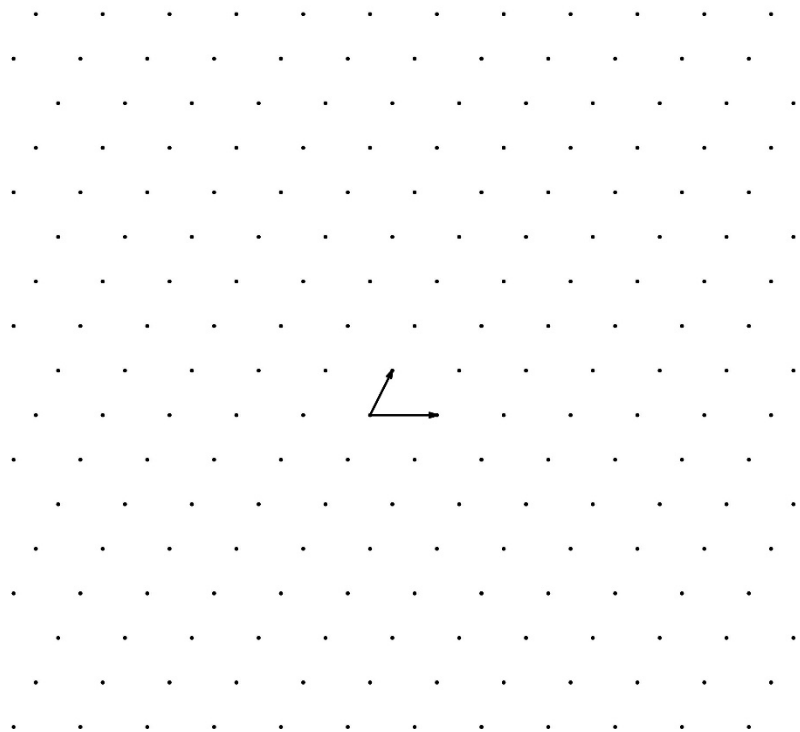


The paralleloid with vertices $\vec{0}$, $\vec{a}$, $\vec{b}$, $\vec{a}+\vec{b}$ is determined up to similarity transformations by the ratio $\dfrac{\|\vec{a}\|}{\|\vec{b}\|}$ of the norms of the two vectors and the angle formed by the vectors. When the ratio is replaced by the inverse (reciprocal) number or the angle θ is replaced by the supplementary angle $\pi-\theta$, the paralleloids are similar.

The lattice is formed by the vertices when the plane is tiled by paralleloids. Then different paralleloids determine the same lattice.

If the pair of the vectors $\vec{a}$ and $\vec{b}$ and the pair of the vectors $\vec{a}'$ and $\vec{b}'$ determine the same lattice,
then there exist integers s, t, u and v such that

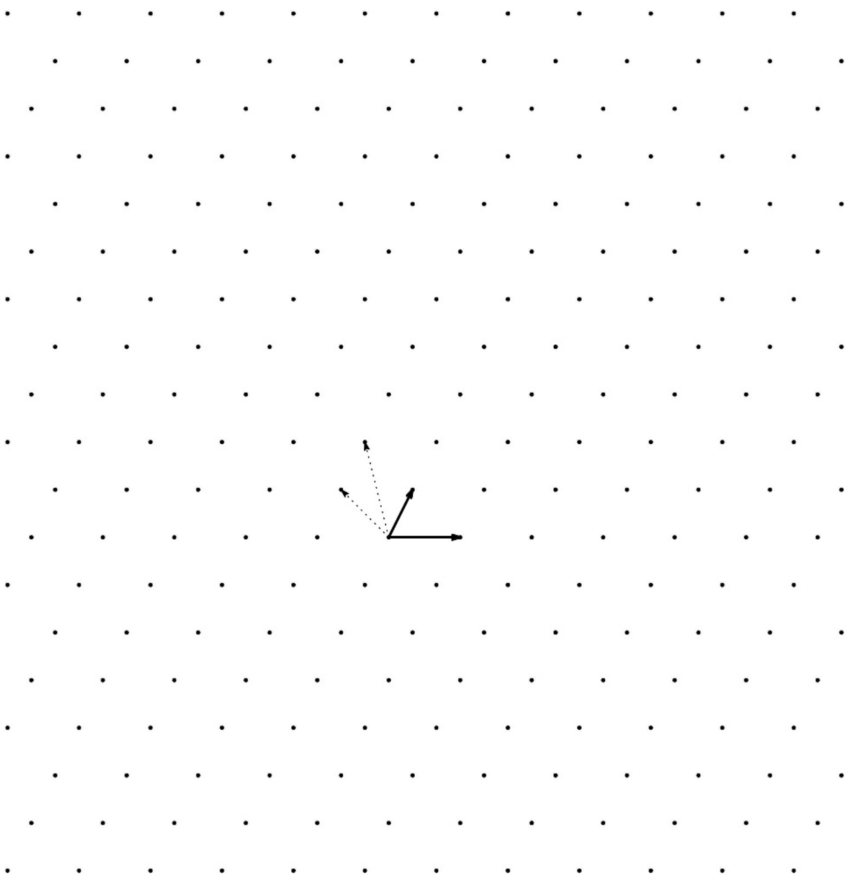
$$\vec{a}' = s\vec{a} + t\vec{b},$$

$$\vec{b}' = u\vec{a} + v\vec{b}.$$

There also exist integers s', t', u' and v' such that

$$\vec{a} = s'\vec{a}' + t'\vec{b}',$$

$$\vec{b} = u'\vec{a}' + v'\vec{b}'.$$



Then

$$\vec{a} = s'(s\vec{a} + t\vec{b}) + t'(u\vec{a} + v\vec{b}) = (s's + t'u)\vec{a} + (s't + t'v)\vec{b},$$
$$\vec{b} = u'(s\vec{a} + t\vec{b}) + v'(u\vec{a} + v\vec{b}) = (u's + v'u)\vec{a} + (u't + v'v)\vec{b}.$$

Therefore

$$\begin{pmatrix} s' & t' \\ u' & v' \end{pmatrix} \begin{pmatrix} s & t \\ u & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Since the matrices are integer matrices, their determinants are integers. By taking the determinants of the both sides of the equality, the determinants of these matrices are ± 1.

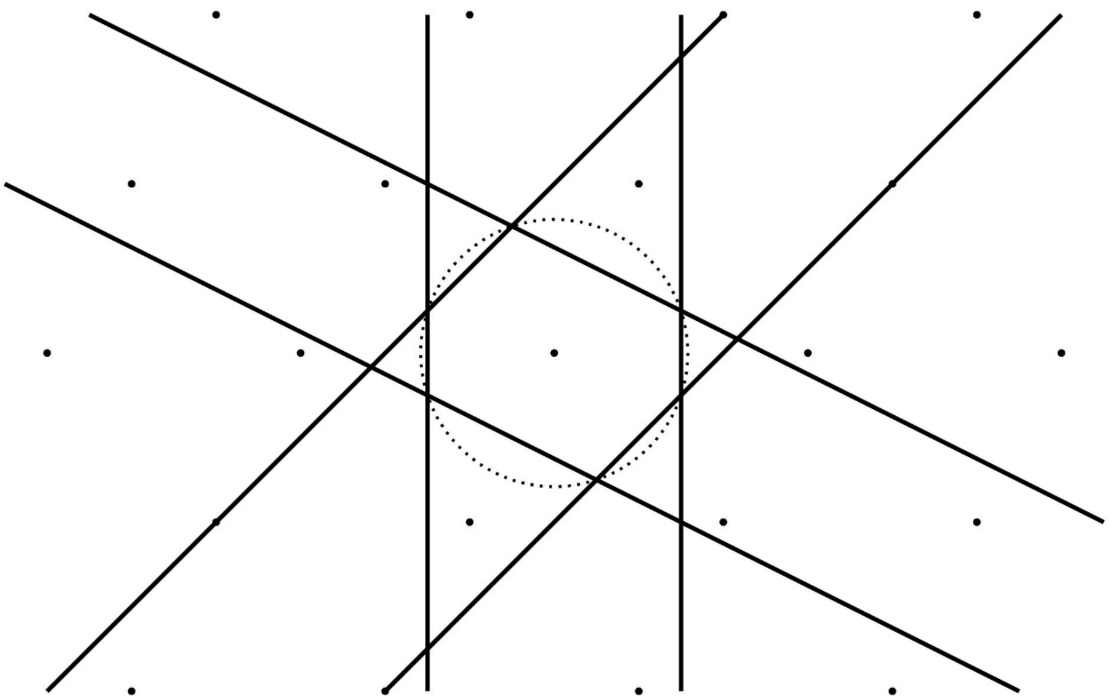
As a conclusion, for an integer matrix $\begin{pmatrix} s & t \\ u & v \end{pmatrix}$ of determinant ± 1, the pairs $\{\vec{a}, \vec{b}\}$ and $\{s\vec{a} + t\vec{b}, u\vec{a} + v\vec{b}\}$ determine the same lattice L.

It is an important fact and we will use it later, however, it does not give us a good geometric idea.

Thus it is nice to consider a figure determined naturally by the lattice L. In fact, there is determined the Dirichelet domain or the Voronoi domain of the lattice.

The figure K in question is the set of such points in the plane that the origin is nearest among the lattice points.

$$K = \{x \in R^2 \mid \forall \ell \in L,\ \|x\| \leq \|x - \ell\|\}.$$



K is either a rectangle or a hexagon symmetric with respect to the center with parallel opposite sides which inscribes a circle.

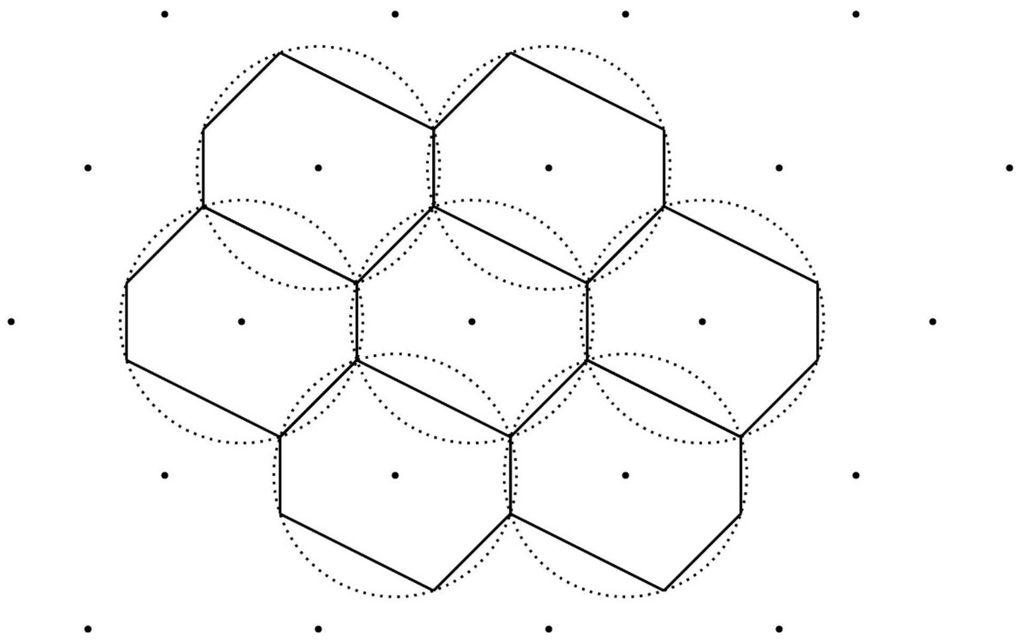
A special case among hexagons is the regular hexagon.

A rectangle can be seen as the limit as the length of a pair of sides tends to 0.

- Let $\vec{p}$ be the point nearest to $\vec{0}$ among the points of the lattice L other than $\vec{0}$.
- Let $\vec{q}$ be the point nearest to $\vec{0}$ among the points of the lattice L other than integer multiples of $\vec{p}$.
- When $\vec{p}$ and $\vec{q}$ are orthogonal, the perpendicular bisector of the segment joining $\vec{0}$ and $\pm\vec{p}$ and that of the segment joining $\vec{0}$ and $\pm\vec{q}$ encloses a rectangle, which is the Dirichelet domain.

- When $\vec{p}$ and $\vec{q}$ are not orthogonal, a vector $\vec{r}$ of the form $\vec{q} \pm \vec{p}$ is the second nearest vector to the origin $\vec{0}$ after $\pm\vec{q}$ among the lattice points L other than integer multiples of $\vec{p}$.
- Then the perpendicular bisector of the segment joining $\vec{0}$ and $\pm\vec{p}$, that of the segment joining $\vec{0}$ and $\pm\vec{q}$, and that of the segment joining $\vec{0}$ and $\pm\vec{r}$ determine a hexagon symmetric with respect to the origin.
- This is the Dirichelet domain of the origin $\vec{0}$. By choosing appropriate signs, the vertices of the hexagon are the circumcenters of the triangles with vertices $\vec{0}$, $\pm\vec{p}$ and $\pm\vec{q}$, and hence the hexagon inscribes a circle.

We can cover the plane periodically using K as a tile.



- By translation by an element of the lattice $L = \{m\vec{a} + n\vec{b} \mid m, n \in Z\}$, the Dirichelet domain of the origin is mapped onto the Dirichelet domain of another lattice point.
- The group L generated by translations in two directions acts on the plane and the orbit space is obtained from K by pasting the opposite sides.
- The orbit space is the same as the space obtained from paralleloid by pasting the opposite sides and it is homeomorphic to a torus.

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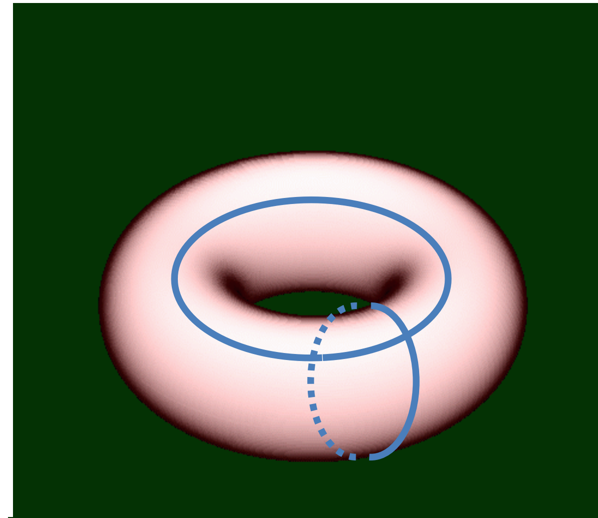
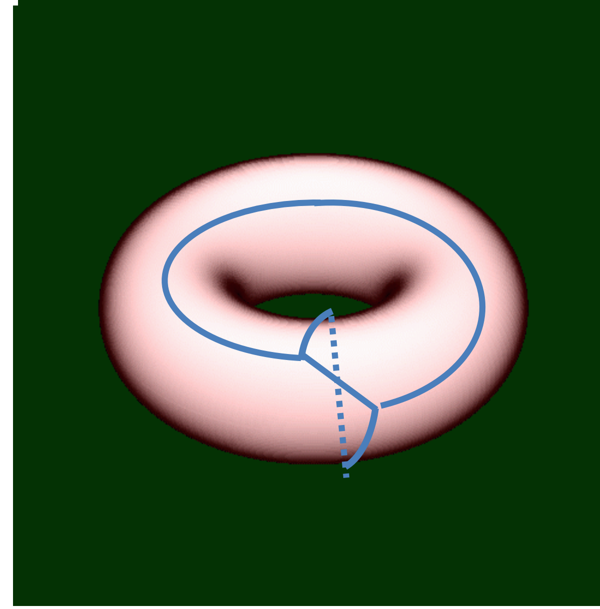
We see that the Euler number of the torus is 0 by dividing it into polygons.

The Euler number of a compact surface is defined to be

$$v - e + f\,,$$

where v , e and f are numbers of vertices, of edges and of faces, respectively, of its cell subdivision.





3. Lattices with symmetry

The lattice L is symmetric with respect to the origin as well as to the mid points of two lattice points, however, only several special lattices have symmetry with respect to a line or rotational symmetry.

If the lattice L is mapped to itself by a move $\vec{x} \longmapsto A\vec{x} + \vec{c}$, then $\vec{0} \longmapsto A\vec{0} + \vec{c} = \vec{c} \in L$. Here, A is an orthogonal matrix.
Since the translation by $\pm\vec{c}$ maps the lattice L to itself, $\vec{x} \longmapsto A\vec{x}$ maps the lattice L to itself.

Then the vectors $\vec{a}$ and $\vec{b}$ generating the lattice are mapped to $A\vec{a} = s\vec{a} + t\vec{a}$ and $A\vec{b} = u\vec{a} + v\vec{a}$, respectively.

Since this can be written as $A \begin{pmatrix} \vec{a} & \vec{b} \end{pmatrix} = \begin{pmatrix} \vec{a} & \vec{b} \end{pmatrix} \begin{pmatrix} s & u \\ t & v \end{pmatrix}$,

for $P = \begin{pmatrix} \vec{a} & \vec{b} \end{pmatrix}$,

$$A = P \begin{pmatrix} s & u \\ t & v \end{pmatrix} P^{-1}.$$

In particular, the trace $\mathrm{Tr}\, A = \mathrm{Tr} \begin{pmatrix} s & u \\ t & v \end{pmatrix} = s + v$ is an integer.

The trace of the orthogonal matrix A is 0 if A is a reflection and is $2\cos\theta$ if A is the rotation by the angle θ. Since it is an integer, it takes one of the values -2, -1, 0, 1 or 2.

If A is a reflection, we can find points of L on the axis of the symmetry, and we see that the fundamental domain of lattice L is either a rectangle or a rhombus (a diamond).

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If A is a reflection, we can find points of L on the axis of the symmetry, and we see that the fundamental domain of lattice L is either a rectangle or a rhombus (a diamond).

Thus, if A is a rotation by the angle θ, then θ is equal to one of the following angles:

$$\pi;\ \pm\frac{2}{3}\pi;\ \pm\frac{1}{2}\pi;\ \pm\frac{1}{3}\pi;\ 0.$$

This restriction on the angles of the rotational symmetries of lattices explains the frequent appearence of “stoicheia”.

4. Orbit spaces

Let F be a figure on the Euclidean plane R^2. Assume that the isometry group (congruence group) $G = I(F)$ contains a lattice L generated by two translations by linearly independent vectors.

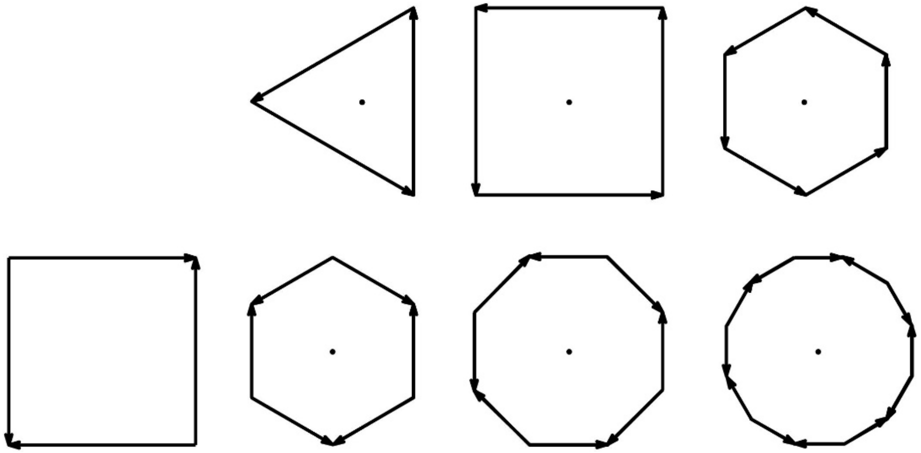
Definition. For a point $x \in R^2$, put

$G_x = \{g \in G \mid g(x) = x\}$.

G_x is called the isotropy subgroup at x.

G_x is a subgroup of the group $\mathrm{O}(2)$ consisting of 2×2 orthogonal matrices. When G contains a lattice L, elements of G_x preserve the lattice$L$ by the argument above. Thus G_x is isomorphic to one of the following groups:

- $\{\mathrm{id}\}$ (the group consisting of only one element)
- C_2 (symmetry w.r.t. a point), C_3, C_4, C_6
- D_2 (symmetry w.r.t. a line), D_4, D_6, D_8, D_{12}

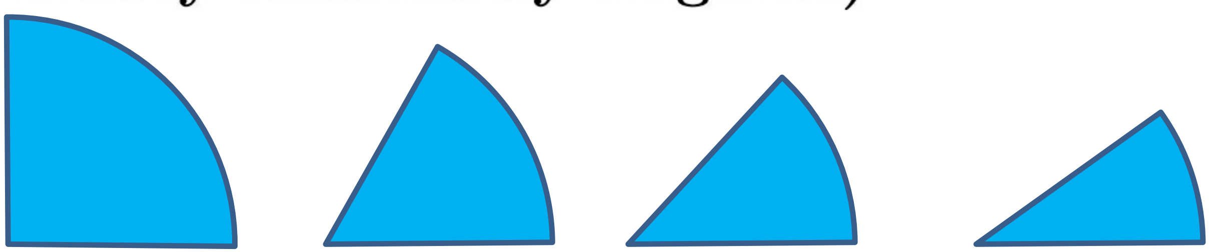


Thus a neighborhood of a point $[x]$ in the orbit space R^2/G is isometric (congruent) to one of the followings:

- The origin of the plane.
- The vertex of a cone of angle π, $\dfrac{2\pi}{3}$, $\dfrac{\pi}{2}$ or $\dfrac{\pi}{3}$.
- The origin of the half space $\{(x, y) \mid x \leqq 0\}$.
- The vertex of a sector of angle $\dfrac{\pi}{2}$, $\dfrac{\pi}{3}$, $\dfrac{\pi}{4}$ or $\dfrac{\pi}{6}$.

5. Flat orbifolds of dimension 2

- The figure R^2/G is called a flat orbifold of dimension 2.
- A flat orbifold of dimension 2 is a space where a neighborhood of each point is one of the following forms: an open set in the plane; an open set in the half plane; a cone of angle $\dfrac{2\pi}{n}$ for a positive integer n: a sector of angle $\dfrac{\pi}{n}$ an open set in the plane. (It is locally realized by origami.)

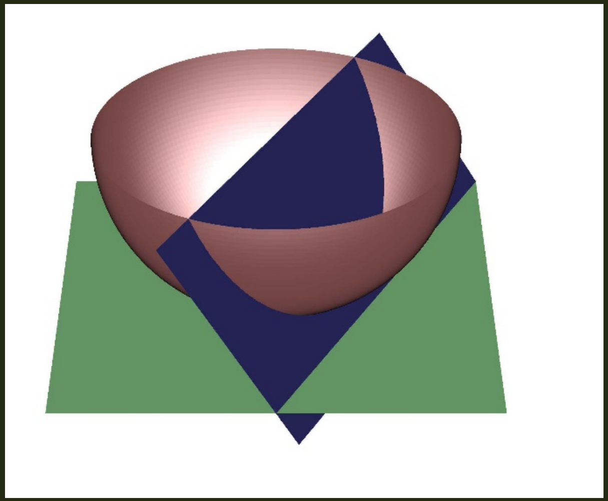


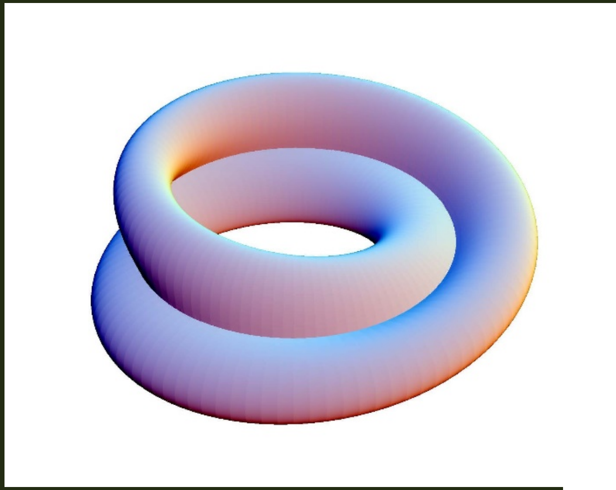
- For a flat orbifold of dimension 2 with boundary edges (and the vertices of sectors), we can construct its double by attaching two copies along the edges and the double is a flat orbifold of dimension 2 without boundary edges (but possibly with vertices of cones).

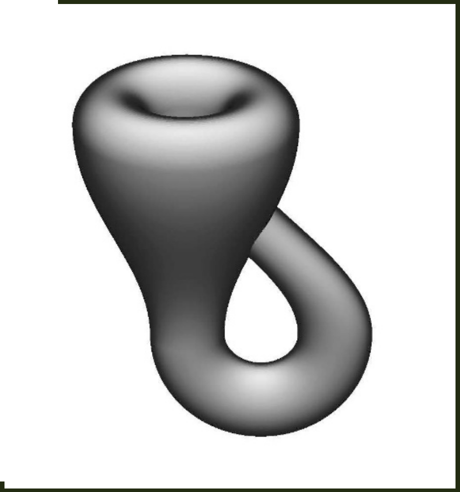
- First we classify flat orbifolds of dimension 2 without boundary edges (but possibly with vertices of cones).

- Then by finding an axis of symmetry and by taking the half, we can classify flat orbifold of dimension 2 with boundary edges.

- There are seven compact flat orbifolds of dimension 2 without boundary edges (but possibly with vertices of cones) . They are T^2, Kl, $RP^2(2,2)$, $S^2(2,3,6)$, $S^2(2,4,4)$, $S^2(3,3,3)$, $S^2(2,2,2,2)$.
- Here T^2, Kl, RP^2, S^2 are the torus, the Klein bottle, the projective plane and the sphere, respectively. The numbers (n_1, n_2), (n_1, n_2, n_3), (n_1, n_2, n_3, n_4) mean that the space has the vertices of cone of angle $\dfrac{2\pi}{n_1}, \ldots$.
- The fact that there are only seven compact flat orbifolds of dimension 2 without boundary edges follows from that their orbifold Euler number is zero.







- **Flat orbifolds of dimension 2 without boundary edges (but possibly with vertices of cones) are topologically a closed surface because a vertex of a cone also has a neighborhood homeomorphic to a disk.**
- **Closed surfaces are classified by orientability and the Euler number.**
- **The Euler number $\chi(S)$ of a closed surface S is defined to be**
  (the number of vertices) − (the number of edges)
  + (the number of faces)
  after taking a triangulation of the surface.

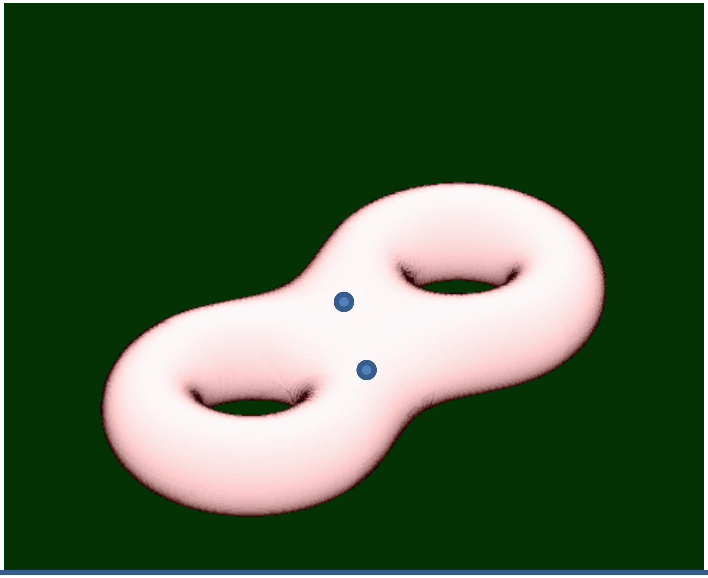
- The Euler number of closed orientable surfaces takes values 2, 0, -2, -4, ... The corresponding surfaces are the sphere S^2, the torus T^2, the closed orientble surface of genus 2, that of genus 3, ...

- The Euler number of closed nonorientable surfaces takes values 1, 0, -1, -2, ... The corresponding surfaces are the projective plane RP^2, the Klein bottle Kl, the closed nonorientble surface of genus 3, that of genus 4, ...

6. Euler numbers and branched coverings

- For a k fold covering map $S_k \longrightarrow S_1$ without branching points between closed surfaces, the equality $\chi(S_k) = k\,\chi(S_1)$ holds.
- This is just because the numbers of vertices, edges, faces of S_k are k times those of vertices, edges, faces of S_1, respectively.
- For flat orbifolds of dimension 2 without boundary edges, it is natural to consider branched covering maps between them.
- A branched covering map locally allows the projections from a disk or a cone to the cone which is the orbit space of a rotational action by the cyclic group C_m of order m.

- **A flat orbifolds of dimension 2 without boundary edges is topologically a closed surface S with finitely many cone points $p_1, \ldots, p_k$ of cone angles $\dfrac{2\pi}{n_1}, \ldots, \dfrac{2\pi}{n_k}$, respectively ($n_1 \geqq 2, \ldots, n_k \geqq 2$). Let $S(n_1, \ldots, n_{\mathrm{k}})$ denote the orbifold.**



Now we define the Euler number $\chi(S(n_1,\dots,n_k))$ of the flat orbifold $S(n_1,\dots,n_k)$ of dimension 2.

- We triangulate the surface so that the cone points are vertices. The cone point of angle $\dfrac{2\pi}{n}$ should be counted as 1 vertex after taking n fold branched cover. Hence it should be counted as $\dfrac{1}{n}$ vertex.
- Thus we define the Euler number $\chi(S(n_1,\dots,n_k))$ of the flat orbifold $S(n_1,\dots,n_k)$ of dimension 2 by

$$\begin{aligned}\chi(S(n_1,\dots,n_k)) &= \chi(S) - \left(1-\frac{1}{n_1}\right) - \cdots - \left(1-\frac{1}{n_k}\right)\\ &= \chi(S) - k + \left(\frac{1}{n_1} + \cdots + \frac{1}{n_k}\right).\end{aligned}$$

- By this definition, if there is an m fold branced covering map $S_m(n_1,\ldots,n_k) \longrightarrow S_1(n'_1,\ldots,n'_{k'})$, then

$$\chi(S_m(n_1,\ldots,n_k)) = m\ \chi(S_1(n'_1,\ldots,n'_{k'})).$$

- For the flat orbifold $S(n_1,\ldots,n_k)$, there is a branched covering map from the 2 dimensional torus. Hence $\chi(S(n_1,\ldots,n_k)) = 0$.
- Thus to classify flat orbifolds, we enumerate $S(n_1,\ldots,n_k)$ such that

$$\chi(S) - k + \left(\frac{1}{n_1} + \cdots + \frac{1}{n_k}\right) = 0.$$

Proof of the classification of flat orbifolds

- First,

$$\chi(S) = \left(1 - \frac{1}{n_1}\right) + \cdots + \left(1 - \frac{1}{n_k}\right) \geqq 0.$$

- The equality holds only if $k = 0$, and then $S = T^2$ or $S = \mathrm{Kl}$.
- If equality does not hold, either $\chi(S) = 1$ or $\chi(S) = 2$.
- $\chi(S) = 1$ implies that $S = RP^2$ and $\chi(S) = 2$ implies that $S = S^2$.

- Since $\left(1-\frac{1}{n_1}\right) \geqq \frac{1}{2}$, $k \leqq 2\chi(S)$ holds.

- The case where $\chi(S) = 1$:

  For $k = 1$, there are no positive integers n such that $1 = \left(1-\frac{1}{n}\right)$.

  For $k = 2$, the equality $1 = \left(1-\frac{1}{n_1}\right) + \left(1-\frac{1}{n_2}\right)$ holds only if $n_1 = n_2 = 2$.

The case where $\chi(S) = 2$:

- For $k = 1$, there are no positive integers n such that $2 = \left(1 - \dfrac{1}{n}\right)$.
- For $k = 2$, there are no positive integers n_1, n_2 such that $2 = \left(1 - \dfrac{1}{n_1}\right) + \left(1 - \dfrac{1}{n_2}\right)$.

- For $k = 3$,

$$2 = \left(1 - \frac{1}{n_1}\right) + \left(1 - \frac{1}{n_2}\right) + \left(1 - \frac{1}{n_3}\right)$$

is rewritten as

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1.$$

We find the integers n_1, n_2, n_3 $(n_1 \leqq n_2 \leqq n_3)$ satisfying the equation as follows:
$(n_1, n_2, n_3) = (2, 3, 6)$, $(2, 4, 4)$, $(3, 3, 3)$.

- For $k = 4$,

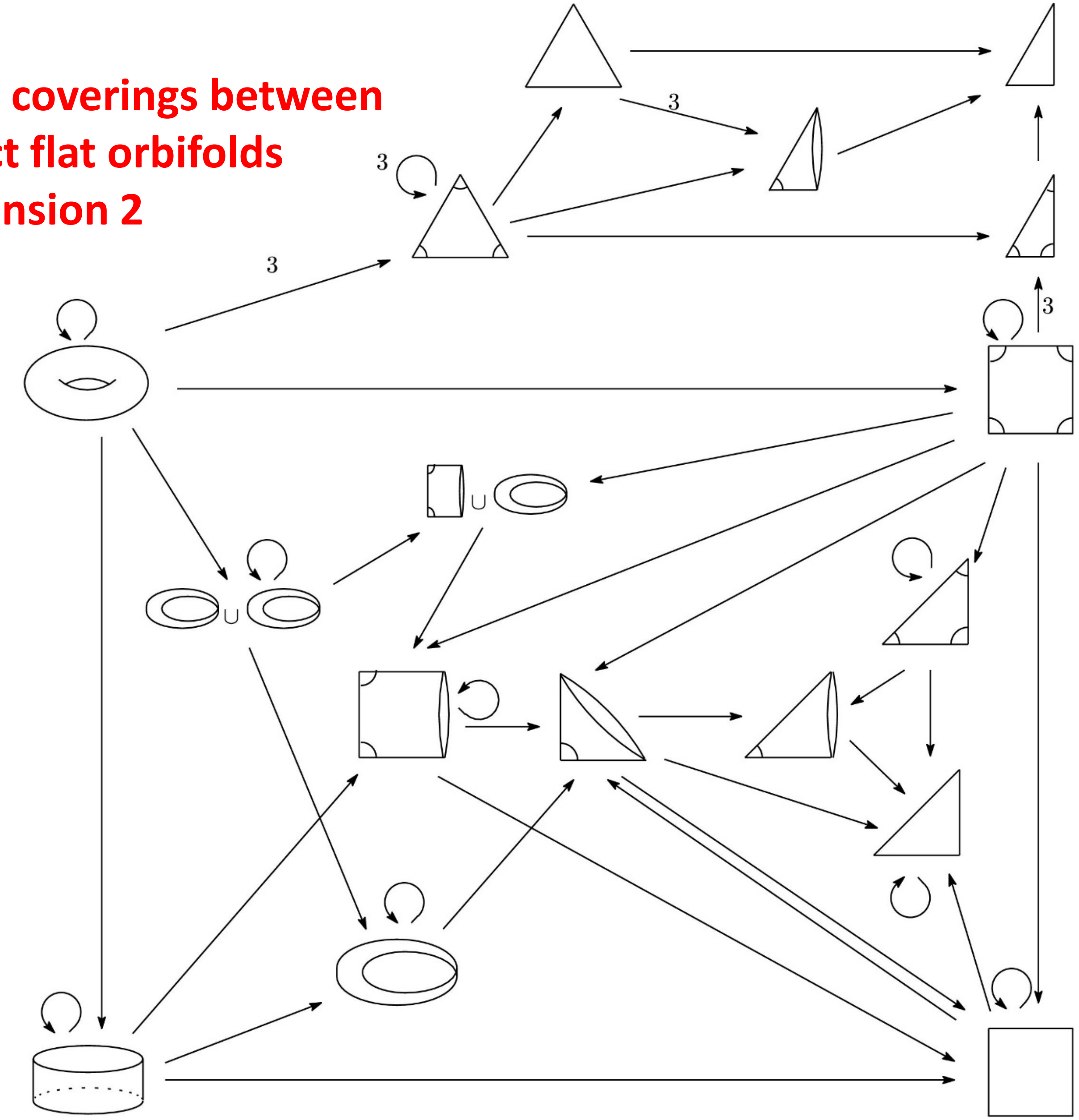
$$2 = \left(1 - \frac{1}{n_1}\right) + \left(1 - \frac{1}{n_2}\right) + \left(1 - \frac{1}{n_3}\right) + \left(1 - \frac{1}{n_4}\right)$$

is satisfied only when $n_1 = n_2 = n_3 = n_4 = 2$.

- Thus we showed that there are seven compact flat orbifolds of dimension 2 without boundary edges. They are T^2, Kl, $RP^2(2,2)$, $S^2(2,3,6)$, $S^2(2,4,4)$, $S^2(3,3,3)$, $S^2(2,2,2,2)$.

- To classify compact flat orbifolds of dimension 2 with boundary edges, we study the existence of axis of symmetry in these seven orbifolds.
- There is 1 pair of axes in T^2, and also 1 pair of axes in Kl. There are no axes in $RP^2(2,2)$. The numbers of axes are 1 in $S^2(2,3,6)$, 2 in $S^2(2,4,4)$, 2 in $S^2(3,3,3)$, 3 in $S^2(2,2,2,2)$.
- Thus there are ten compact flat orbifolds of dimension 2 with boundary edges.

Regular coverings between compact flat orbifolds of dimension 2



The fact that there are 17 types of compact flat orbifolds of dimension 2 is equivalent to that there are 17 types of figures on the plane with periodicity and symmetry.

To understand these 17 types of figures better, you can use the software "KaleidoPaint" by Jeff Weeks:
http://www.geometrygames.org/KaleidoPaint/index.html

You can also visit the following web page:
http://faculty.ms.u-tokyo.ac.jp/users/urabe/urabe/index.html

As a reference which also contains some part of the next lecture, we recommend to look at「結晶群」(Christallographic Groups)（共立講座 数学探検 7）in Japanese by Professor Toshitake Kohno, who will give the last 3 lectures of this series.

You can also find a short text at:
http://tambara.ms.u-tokyo.ac.jp/2011/201109v2ni.pdf