Global Forcus on Knowledge
Mathematical Science Developing from Figures

Periodicity and Symmetry

Space filling polyhedra

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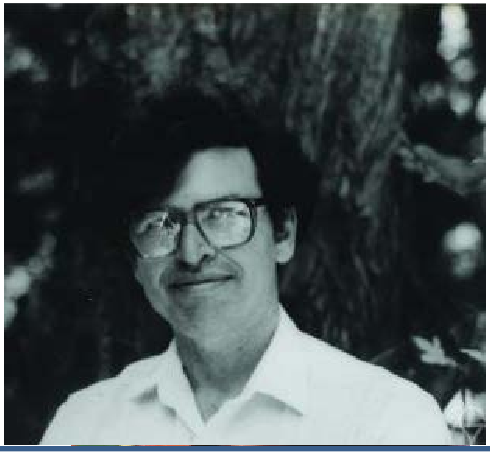
- In the last lecture we explained that there are 17 types of compact flat orbifolds of dimension 2.
- This fact explains that there are 17 types of figures in the plane with periodicity and symmetry.
- In other words, a figure filling the plane is obtained from one of the 17 types of figures by deformation.
- For figures in the 3-dimensional Euclidean space with periodicity and symmetry, we would like to classify the isometry group of the figure.
- A figure has periodicity and symmetry if the isometry group of the figure contains three translations by linearly independent vectors.
- Flat orbifolds of dimension 3 are defined to be the orbit spaces of the actions of isometry groups acting on the 3-dimensional Euclidean space.

- By classifying compact flat orbifolds of dimension 3, we can classify the figures of the 3-dimensional Euclidean space with periodicity and symmetry.
- This classification method is due to Thurston(1946-2012), and the paper of the classification in the 3-dimensional case is published in "Contributions to Algebra and Geometry, Volume 42 (2001), No. 2, 475-507" under the title "On Three-Dimensional Space Groups" by the authors John H. Conway, Olaf Delgado Friedrichs, Daniel H. Huson, and William P. Thurston.
- There are 219 types of 3-dimensional figures with periodicity and symmetry and there are 230 types if we distinguish the mirror images. This was already proven at the end of 19-th century.



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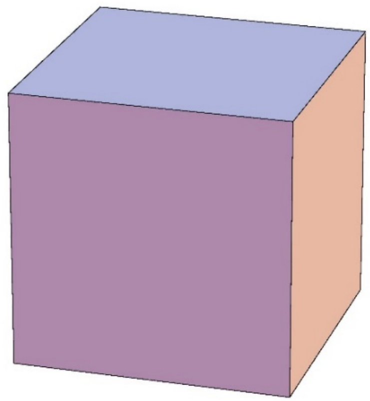
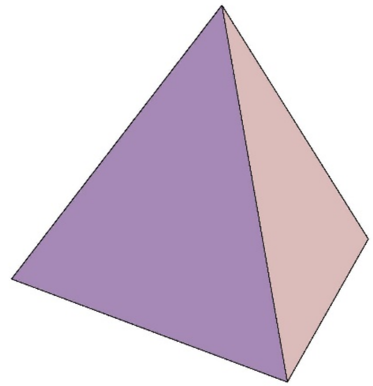
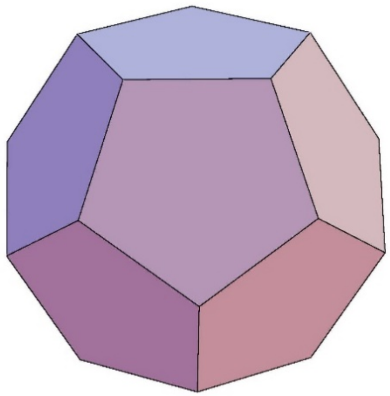
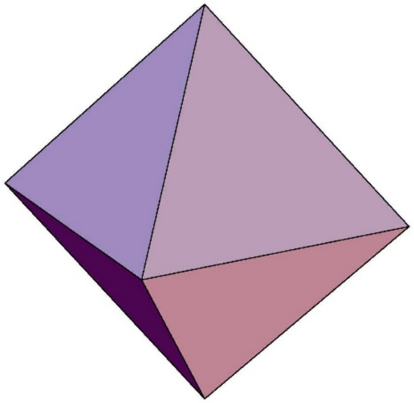


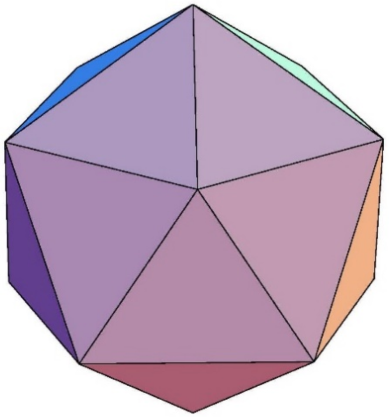




Quiz:

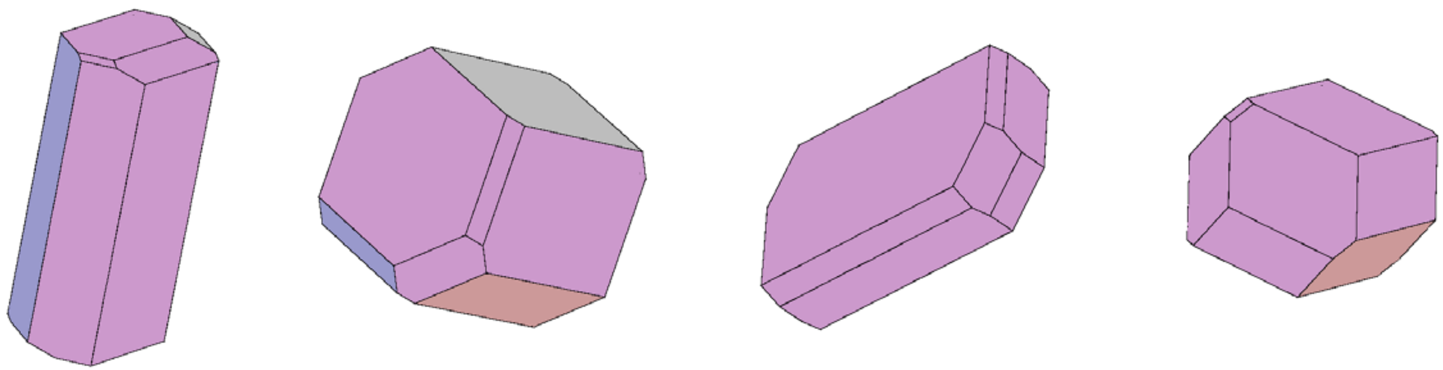
- Can a regular polygon other than cube fill the space by its copies ?
- Is it possible to fill the space using only regular tetrahedra and正 regular octahedra ?





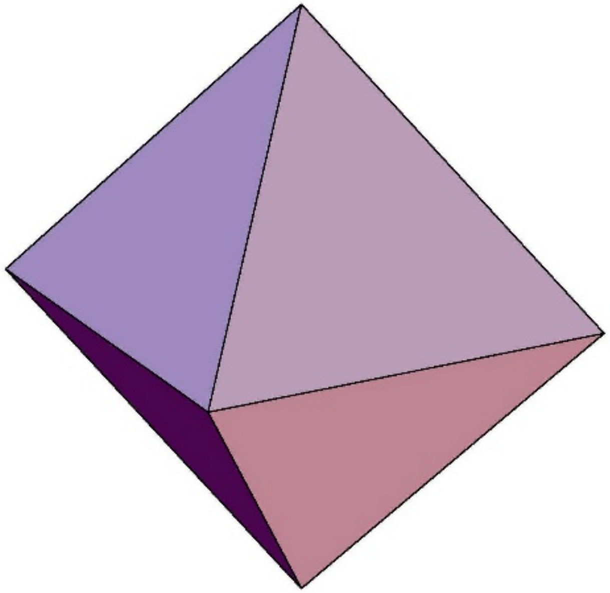
1. Symmetric lattices of dimenssion3

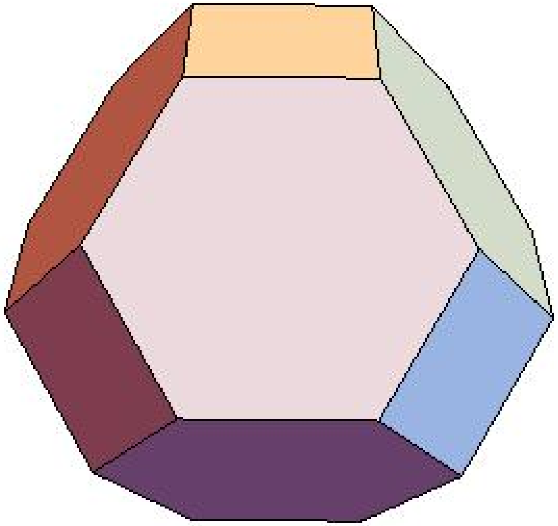
- For the action of the group generated by three translations by linearly independent vectors, the parallelopiped with edges being the three vectors is a fundamental domain. But in most cases, it is not the Dirichelet domain.
- The Dirichelet domain in most cases has 14 faces consisting of 8 hexagons and 6 paralleloids. The configulation is similar to the truncated octahedron.



http://faculty.ms.u-tokyo.ac.jp/users/tsuboi/surijoho/gif/randomfundamentaldomain.html

Here the truncated octahedron is the polyhedron obtained from the regular octahedron by removing six four sided pyramids of edge length one-third around the vertices.





1. Symmetric lattices of dimenssion3

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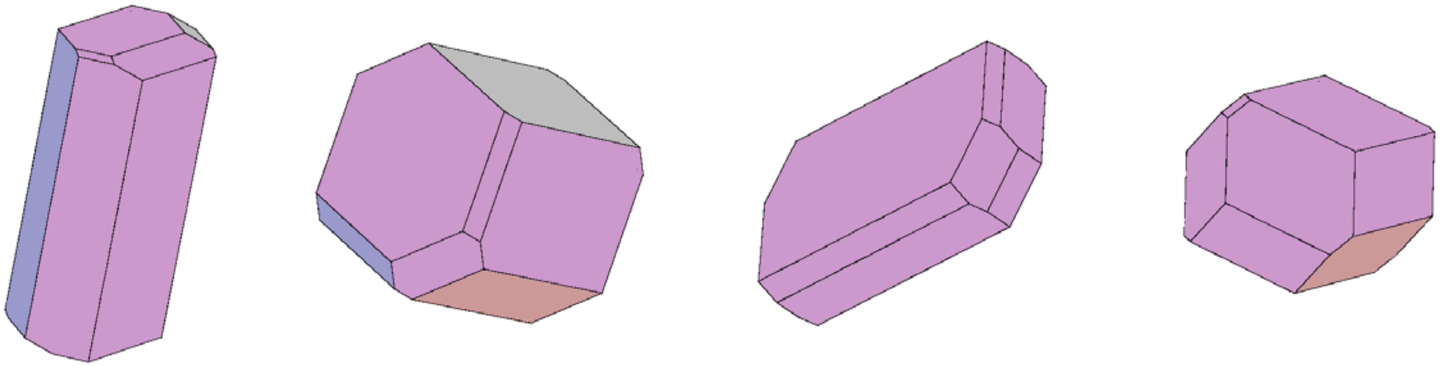

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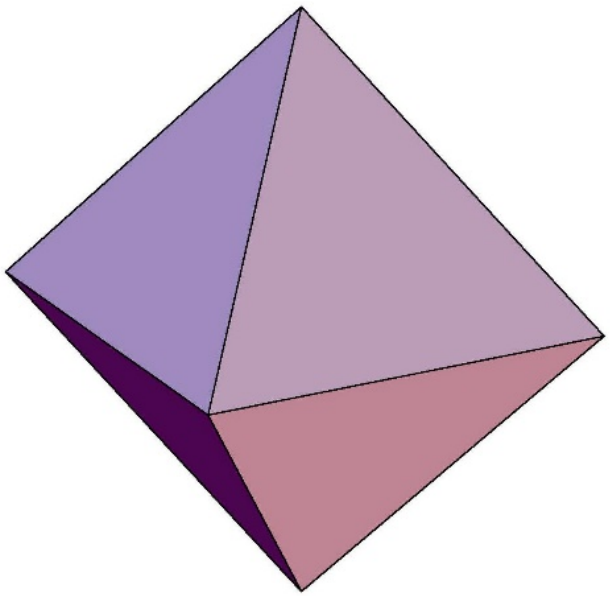

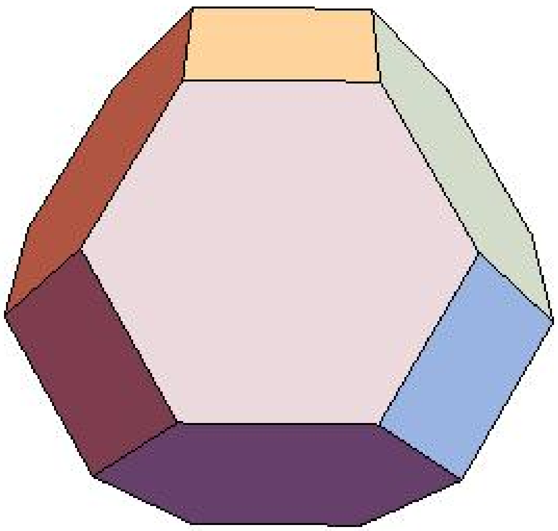

- As for the symmetry of the lattice generated by three linearly independent vectors, our approach to 2-dimensional lattices still works.
- The matrix describing the symmetry is 3×3 orthogonal matrix which is an integer matrix with respect to the basis given by the three generating vectors.

It is known that by choosing an appropriate coordinate system, a 3×3 orthogonal matrix can be written as $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$.

The trace of this matrix is $2\cos\theta \pm 1$. Since it is an integer, $2\cos\theta$ takes one of the following values: $-2, -1, 0, 1, 2.$

Thus θ takes one of the following values: $\pi, \pm\frac{2}{3}\pi, \pm\frac{1}{2}\pi, \pm\frac{1}{3}\pi, 0.$

- **By this argument we will find frequently again "Stoicheia" in the figures.**
- **As for the isotropy group of a point, we find more restrictions.**

For a point $x \in R^3$, let us consider the isotropy subgroup $G_x = \{g \in G \mid g(x) = x\}$.

We look at the orbit space of the action of G_x on the Euclidean space.

In fact, G_x acts on the sphere S^2 of radius 1 centered at x.

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2. Spherical orbifolds of dimension2

The orbit space S^2/G_x is an orbifold of dimension 2. Since it is an orbit space of an action on the sphere, a neighborhood of each point is one of the following forms: an open set in the sphere, an open set of in the hemisphere, a spherical cone of angle $\dfrac{2\pi}{n}$ for a positive integer n, a spherical sector of angle $\dfrac{\pi}{n}$ for a positive integer n.

- **The Euler number of the orbit space is a fraction of 2, the Euler number of the sphere. Note that we defined the Euler number of flat orbifolds of dimension 2, where we only used the angles of cones and sectors, hence the definition is valid for spherical orbifolds of dimension 2.**

Exercise. Following the case of compact flat orbifolds, classify compact spherical orbifolds of dimension 2.

An answer. For the spherical orbifold $S(n_1, \ldots, n_k)$, there would be an appropriate branched covereing map from the sphere. Hence For a positive integer m, the following equality holds: $\chi(S(n_1, \ldots, n_k)) = \dfrac{2}{m}$.

So we look for $S(n_1, \ldots, n_k)$ and m such that

$$\chi(S) - k + \left(\frac{1}{n_1} + \cdots + \frac{1}{n_k}\right) = \frac{2}{m}.$$

We see that

$$\chi(S) = \frac{2}{m} + \left(1 - \frac{1}{n_1}\right) + \cdots + \left(1 - \frac{1}{n_k}\right) = 1 \text{ or } 2.$$

If $\chi(S) = 1$, then $m = 2$ and $k = 0$ or $m = 2\ell$, $k = 1$ and $n_1 = \ell$ (ℓ is an integer greater than 1). These correspond to RP^2 or $RP^2(\ell)$.

If $\chi(S) = 2$, then $k \leq 3$.

If $k = 0$, then the orbifold is S^2.

There are no orbifolds such that $k = 1$.

If $k = 2$, then $m = \ell$ and $n_1 = n_2 = \ell$ (ℓ is an integer greater than 1) and this corresponds to $S^2(\ell, \ell)$.

If $k = 3$, then

$$\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1 + \frac{2}{m}.$$

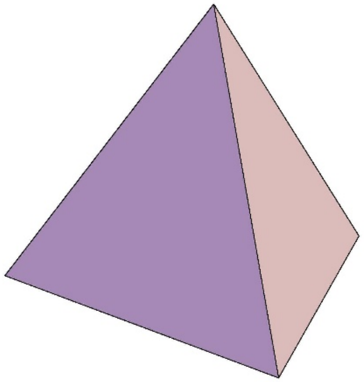
The solutions (n_1, n_2, n_3, m) such that $n_1 \leqq n_2 \leqq n_3$ are $(2, 2, \ell, 2\ell)$ (ℓ is an integer greater than 1), $(2, 3, 3, 12)$, $(2, 3, 4, 24)$, $(2, 3, 5, 60)$.

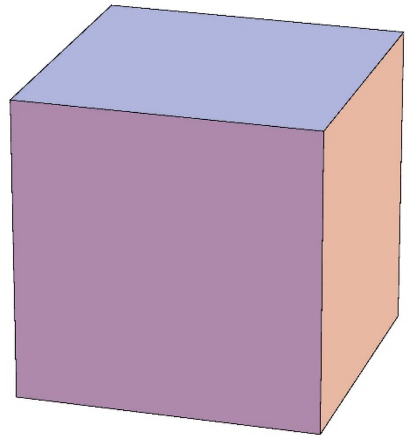
Then the corresponding orbifolds are $S^2(2, 2, \ell)$, $S^2(2, 3, 3)$, $S^2(2, 3, 4)$, $S^2(2, 3, 5)$.

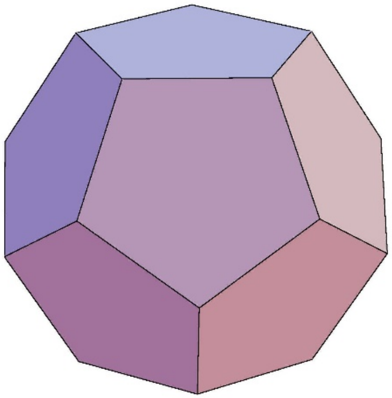
Thus the answer to the exercise is as follows: for an integer ℓ greater than 1, RP^2, $RP^2(\ell)$, $S^2(\ell, \ell)$, $S^2(2, 2, \ell)$, $S^2(2, 3, 3)$, $S^2(2, 3, 4)$, $S^2(2, 3, 5)$.

The isotropy subgroup G_x is

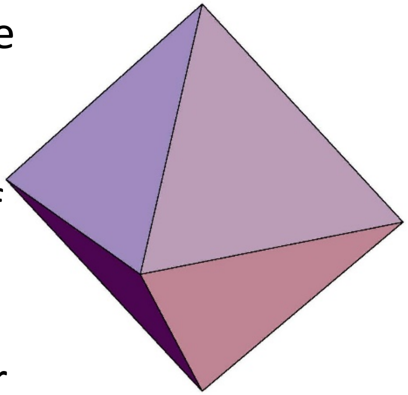
- the group of symmetry with respect to the point x for RP^2,
- the group generated by the rotation by the angle $\dfrac{2\pi}{\ell}$ around the z-axis and the composition of the reflection with respect to the xy-plane and the rotation by the angle $\dfrac{\pi}{\ell}$ around the z-axis for $RP^2(\ell)$,
- the cyclic group C_ℓ generated by the rotation by the angle $\dfrac{2\pi}{\ell}$ for $S^2(\ell, \ell)$,
- the isometry group $D_{2\ell}$ of the regular ℓ-gon for $S^2(2, 2, \ell)$,
- the group of orientation preserving isometries of the regular tetrahedron for $S^2(2, 3, 3)$,
- the group of orientation preserving isometries of the cube or the regular octahedron for $S^2(2, 3, 4)$,
- the group of orientation preserving isometries of the regular dodecahedron or the regular icosahedron for $S^2(2, 3, 5)$.

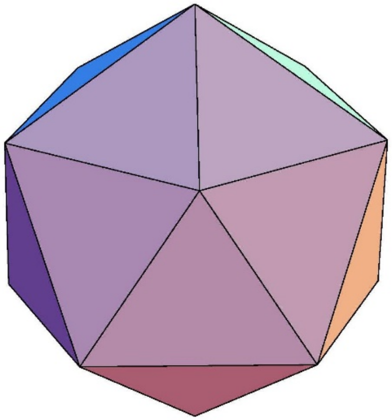




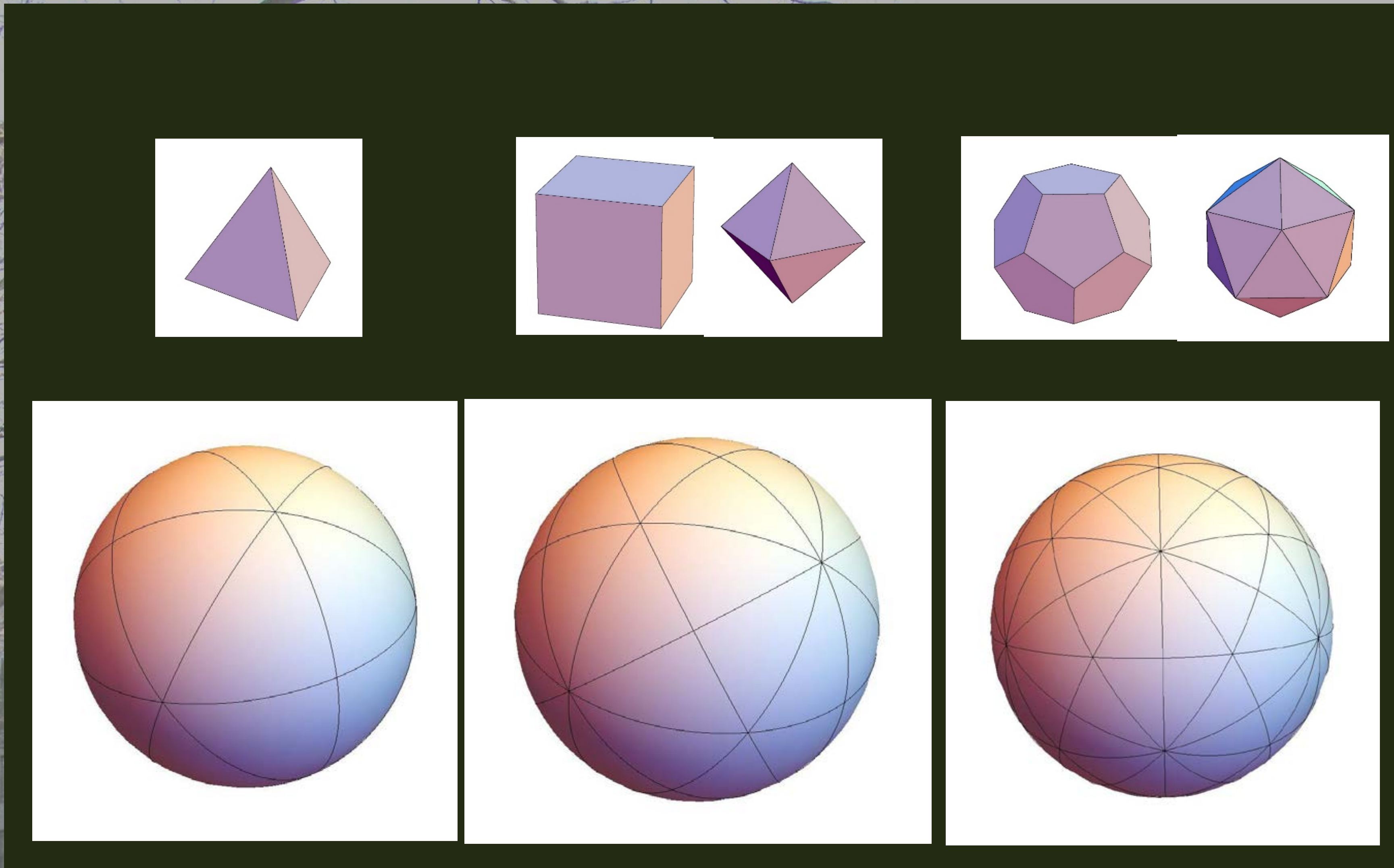


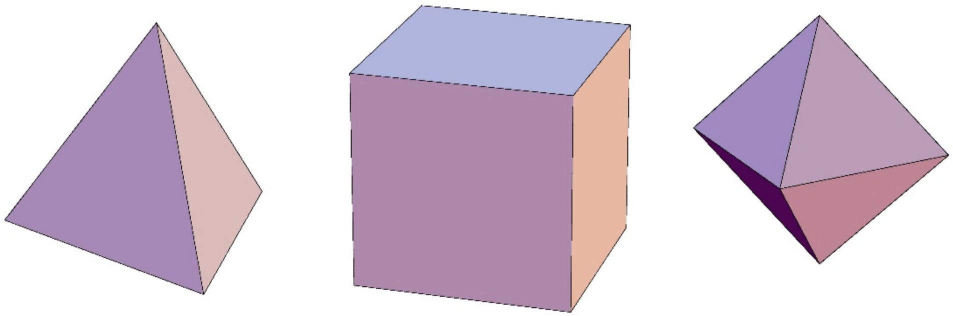
The classification of spherical orbifolds of dimension 2 is closely related to the classification of the regular polyhedra.





Note that the isometry groups of the cube and of the regular octahedron, those of the regular dodecahedron and of the regular icosahedron are isomorphic.





But for the isometry group of the figure in the Euclidean space with periodicity and symmetry, there is a condition on the angles of the rotations, and hence the isometry group of the regular dodecahedron or the regular icosahedron does not appear.

To underestand the action of the group of regular polyhedron, there is a software "KaleidoTile" by Jeff Weeks:
http://www.geometrygames.org/KaleidoTile/index.html

The software GeoGebra http://www.geogebra.org/ supports 3D graphics and we can perform many experiments.

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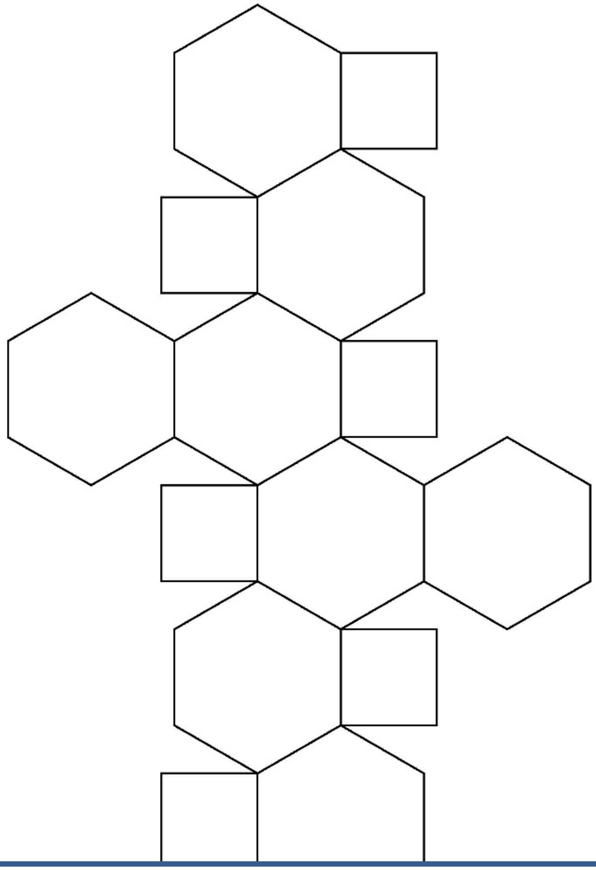
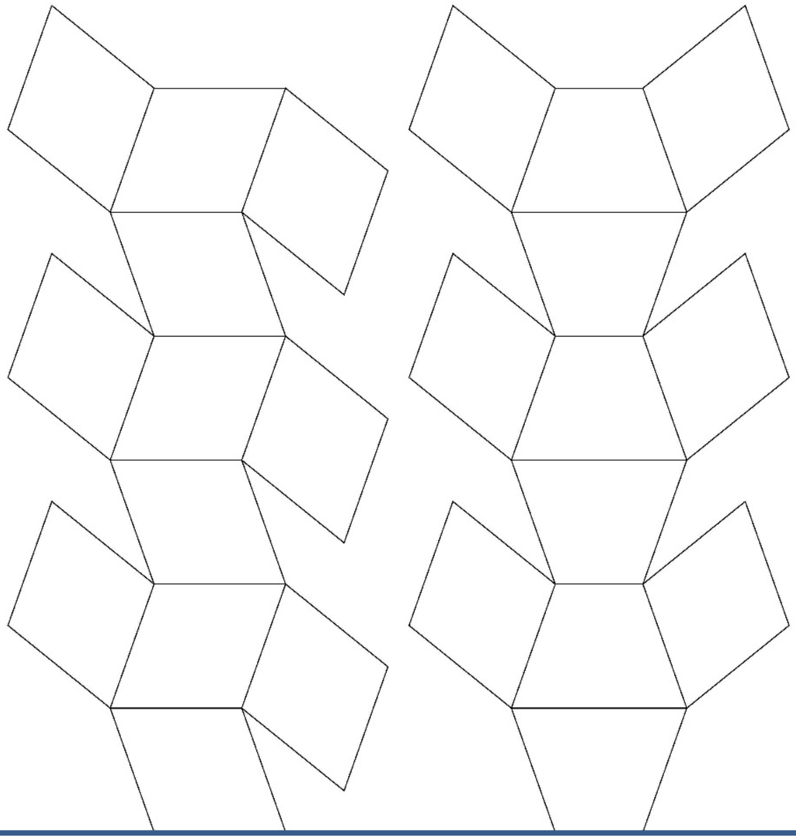
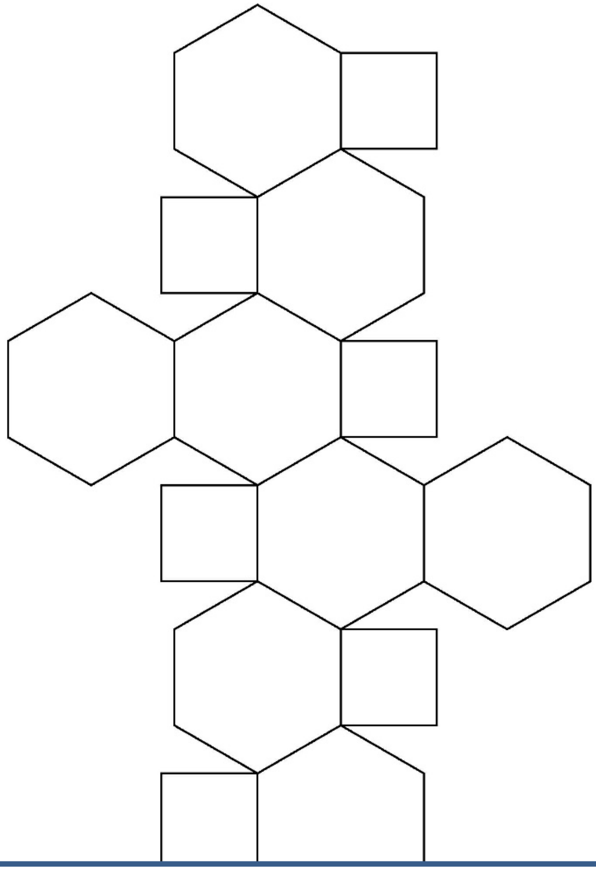
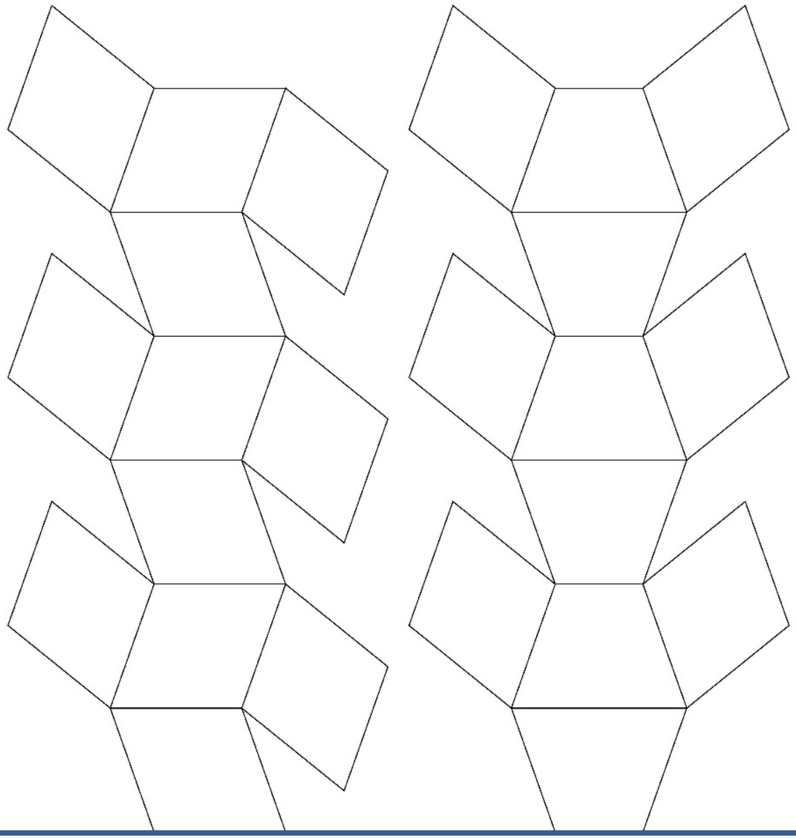
3. Lattices with big symmetry



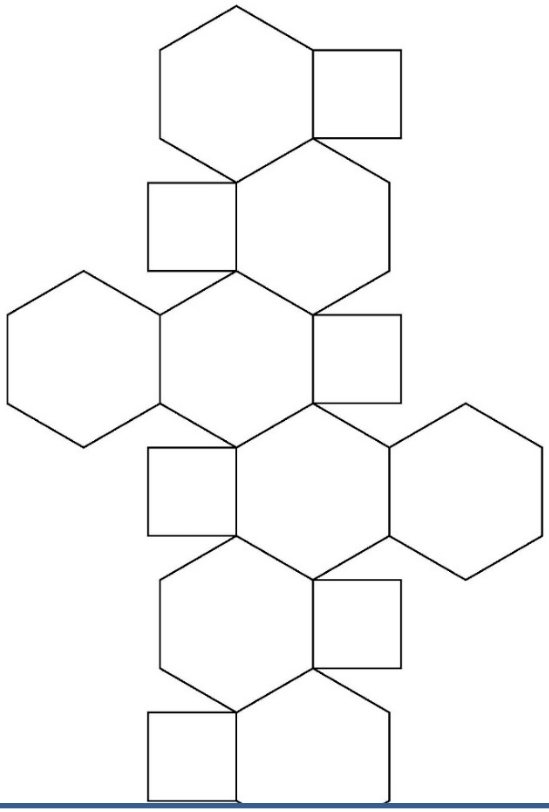


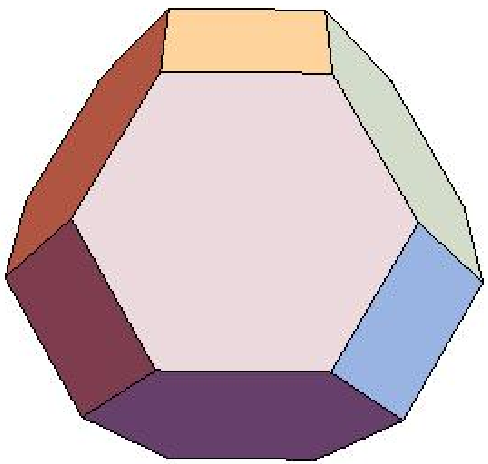
Exercise. Imagine the polyhedral obtained from the following developments.





The polyhedron obtained from the following development is the truncated octahedron.





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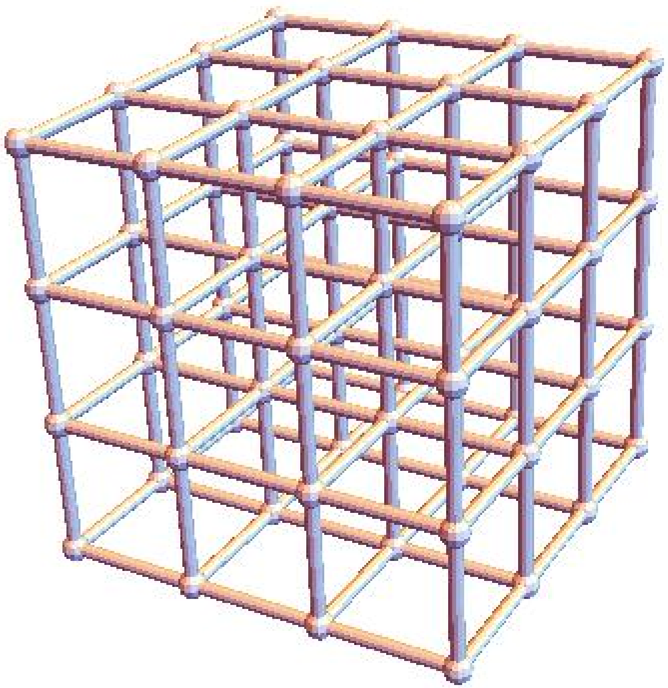
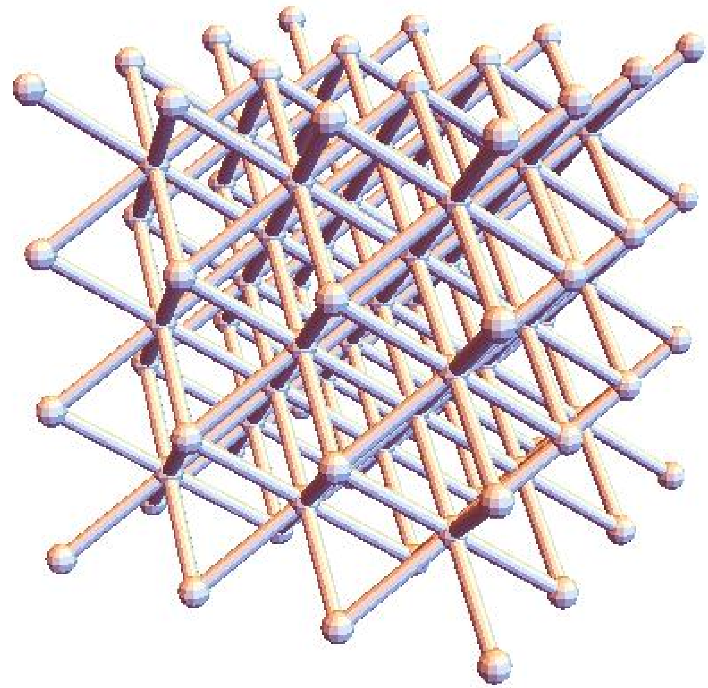
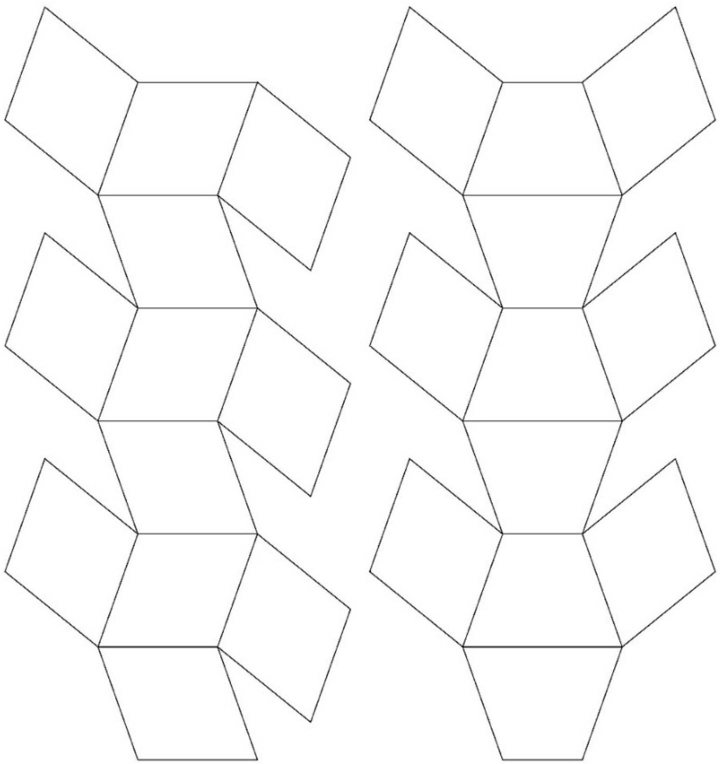
The truncated octahedron is the Dirichelet domain of the body-centered cubic lattice.



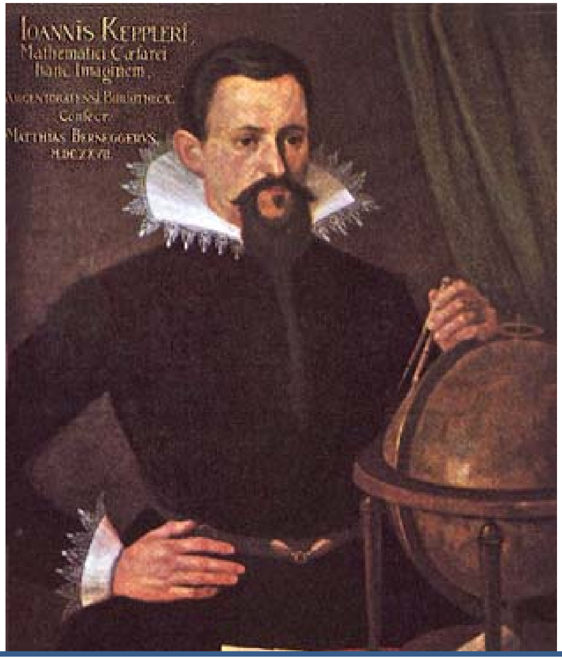


The polyhedral obtained from the following developments are related to the densest packing problem.



The densest packing problem of Kepler:

The density of the densest packing by The balls of the same radius is $\dfrac{\pi}{\sqrt{18}}$.

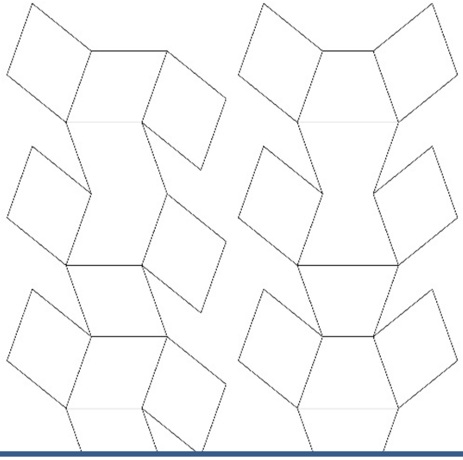


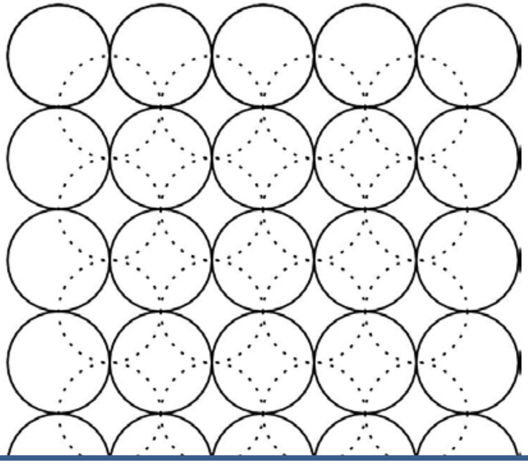
Wikimedia Commons
https://upload.wikimedia.org/wikipedia/commons/d/de/JKepler.png
(ref. 2017/5/18)

We could not distribute marbles, so let us perform a thought experiment.

Exercise with virtual marbles 1. Let make a packing by mables:
the first layer consists of marbles at the square lattice points (circles in solid line);
the second layer consists of marbles at centers of squares of the first layer (circles in dotted line);
The third layer consists of marbles at the same position as the first layer.

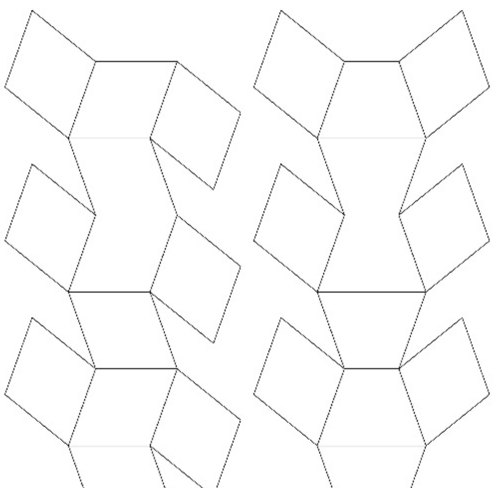
- How many marbles are tangent to a marble ?
- Describe the positions of tangent marbles.

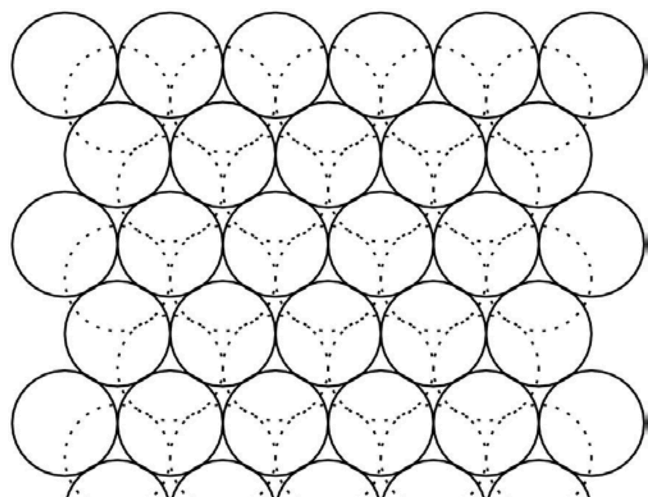




Exercise with virtual marbles 2. Let make a packing by mables:
the first layer consists of marbles at the hexagonal lattice points (circles in solid line);
the second layer consists of marbles at centers of triangles of the first layer (circles in dotted line).

- Observe that there are two diffferent ways to make the third layer.
- How many marbles are tangent to a marble ?
- Describe the positions of tangent marbles.





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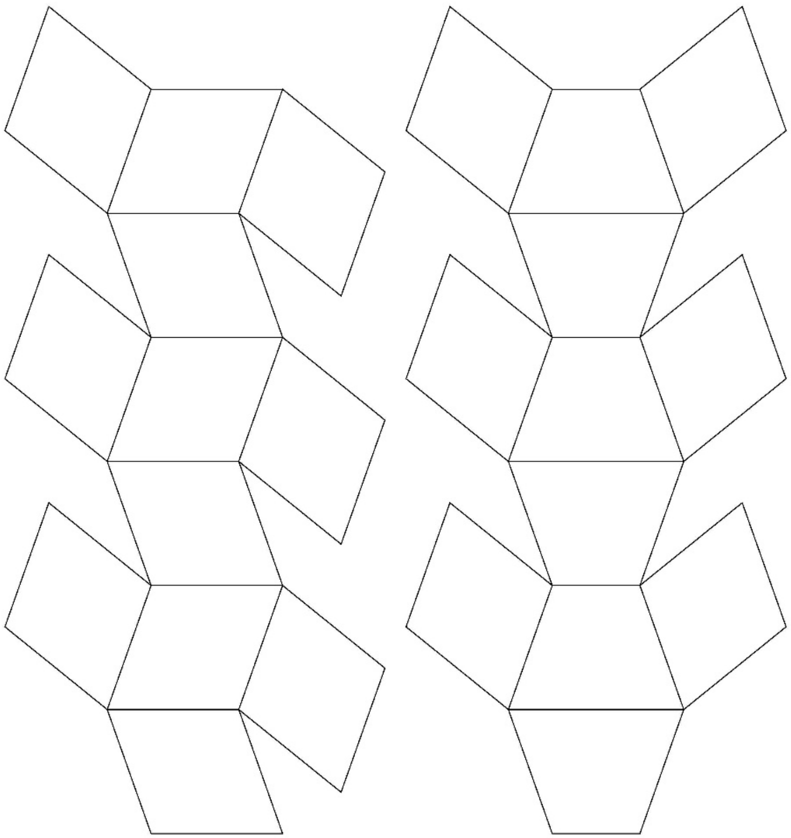
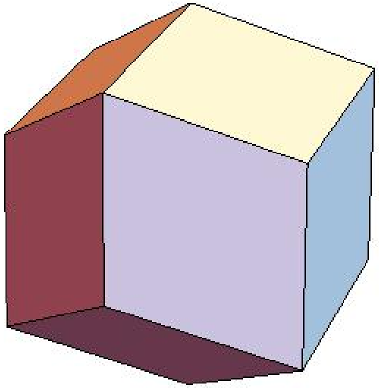


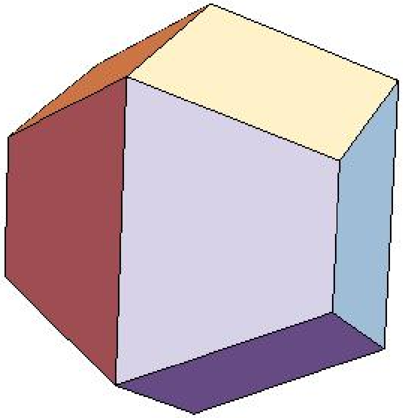
- The development in the left gives the rhombic dodecahedron which is the Dirichelet domain of the packing with layers of square lattices as well as that of one of the packings with layers of hexagonal lattices.

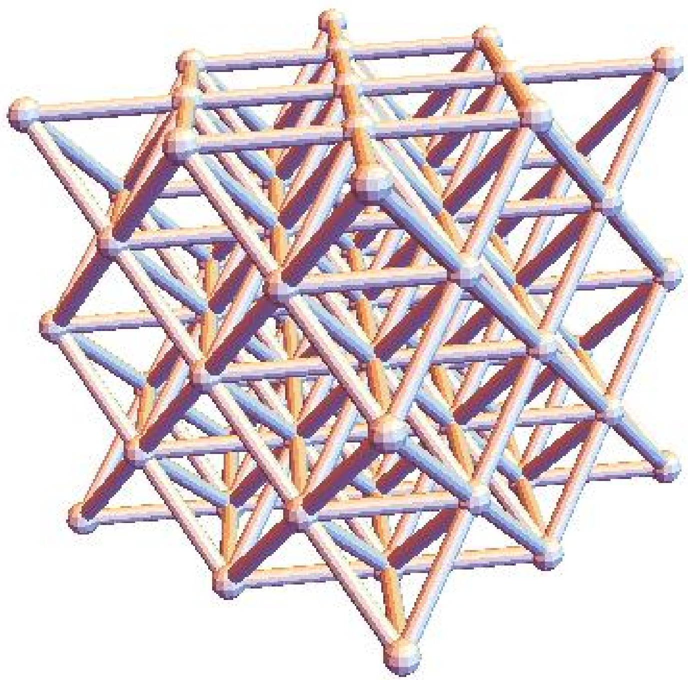
- The development in the right gives the Voronoi domain fo the other packing with layers of hexagonal lattices.

- The rhombic dodecahedron given by the development in the left.



- The polyhedron given by the development in the left.





The Rhombic dodecahedron is the Voronoi domain of the face-centered cubic lattice.

Up to now we treated the Voronoi domains for the lattices generated by translations.

The other polyhedron is the Voronoi domain of the hexagonal closed-packed lattice.

The Dirichelet domains with big symmetries are the cube, the truncated tetrahedron, the rhombic dodecahedron. We may add the hexagonal cylinder.

By Unknown From Wikimedia Commons
(ref. 2016/7/5)

The software “Curved Spaces” by Jeff Weeks gives the idea of the life in the world with periodicity and symmetry:
http://www.geometrygames.org/CurvedSpaces/index.html

4. Periodic figures with big symmetries in dimension 3

Let us consider the graphs such as the cubic lattice, the body-centered cubic lattice, the face centered cubic lattice, etc. with three fold periodicity embedded in the Euclidean space. A graph is the figure consisting of the vertices and edged of length 1. Here after, we call such objects lattices.

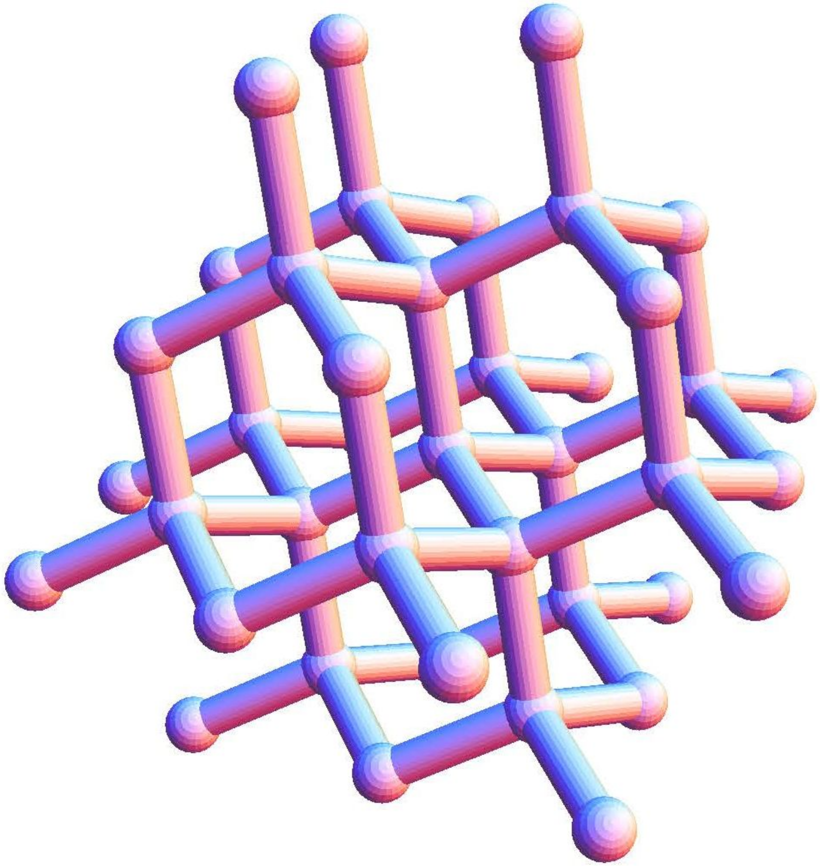
We study the symmetry of such figures.

The case of the biggest symmetry can be formulated as follows:

- at any vertex, there is an isometry changing 2 adjacent edges fixing other adjacent edges, and
- there is an isometry sending each edge to itself reversing the orientation.

Such lattices was studied by Toshikazu Sunada who showed that there are two such lattices: one is the diamond lattice and the other is the K4 lattice.

Toshikazu Sunada, 「Crystals that nature might miss creating」, Notices of the AMS, Volume 55, Number 2, (2008) 208-215.

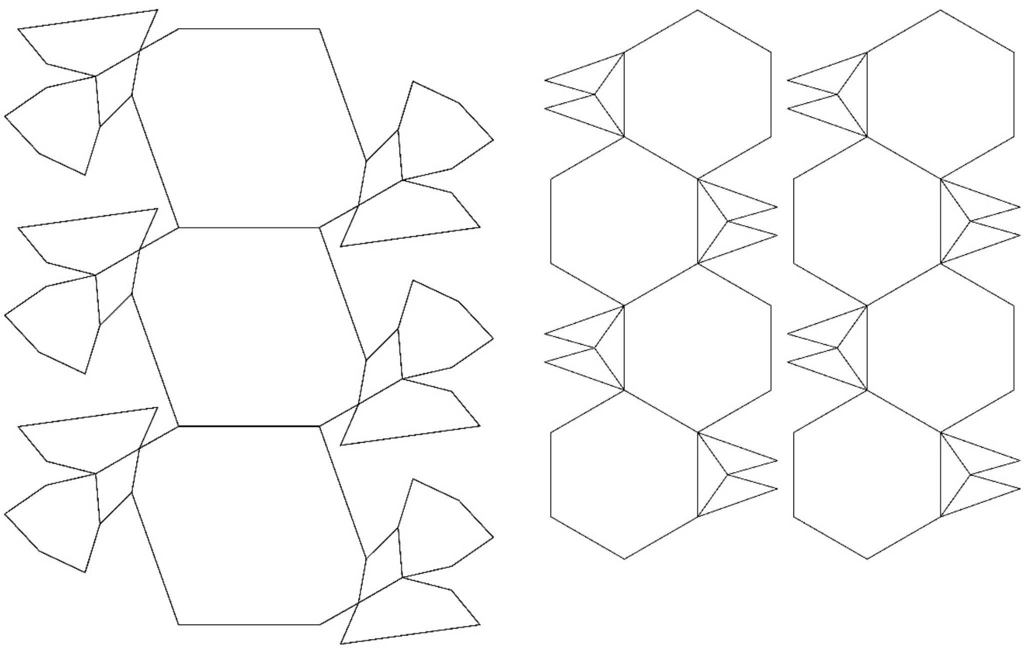


The diamond lattice is as in the figure.

Think about the graph with vertices being those of regular tetrahedron and its barycenter and with edges joining them, and translate the graph by sending the vertices of the tetrahedron to themselves.

Then the configuration around the vertex of the original tetrahedron is the same as that of the barycenter.

Excercise. Make polyhedrons from the distributed developments.
Verify how the space is filled by the polyhedrons
with your neighbors.



In the development in the left, use only one hexagon among the three in the left or in the right.

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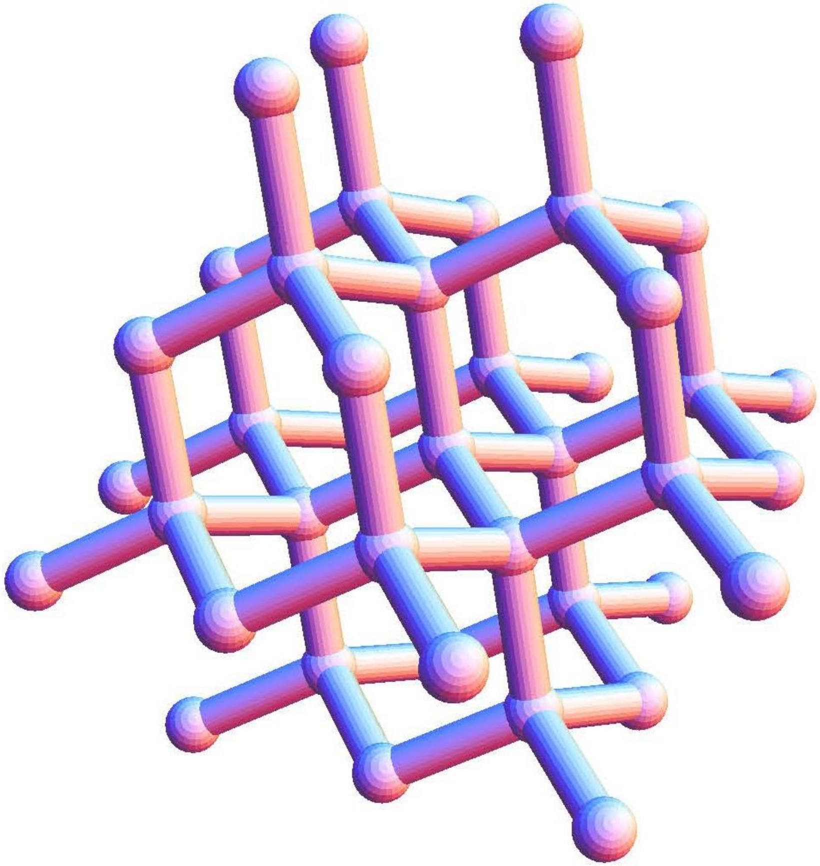

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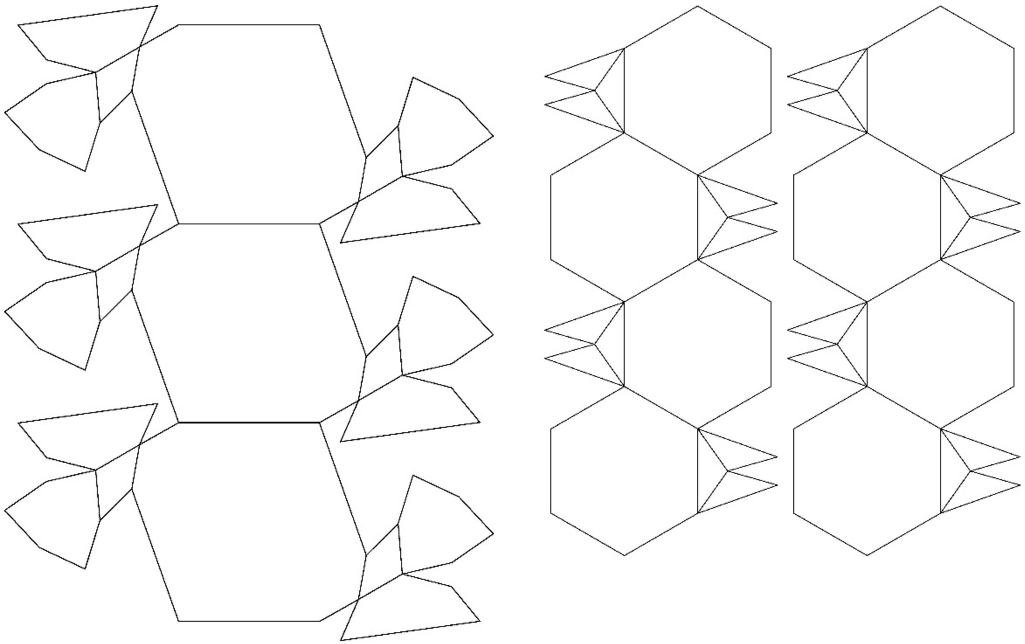
Verify how the space is filled by the polyhedrons with your neighbors.

In the development in the left, use only one hexagon among the three in the left or in the right.



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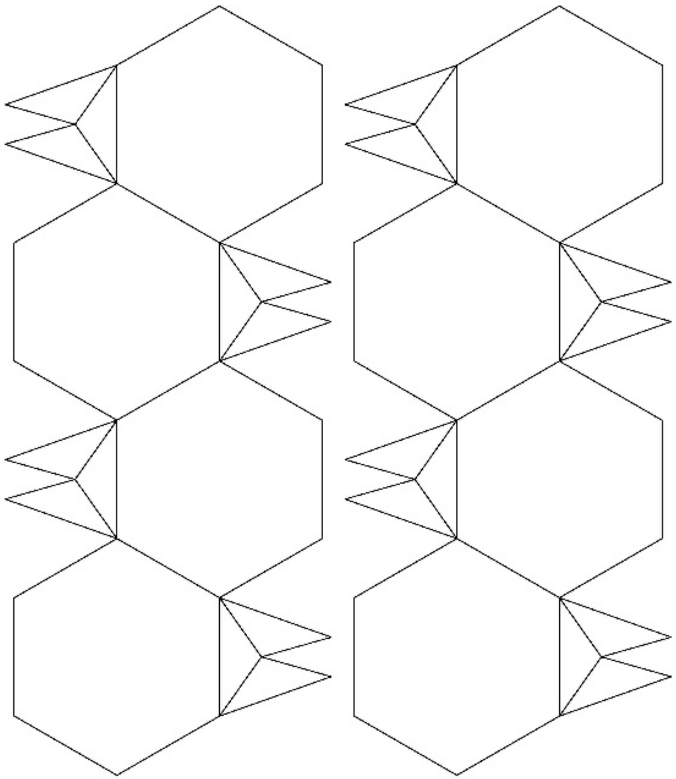
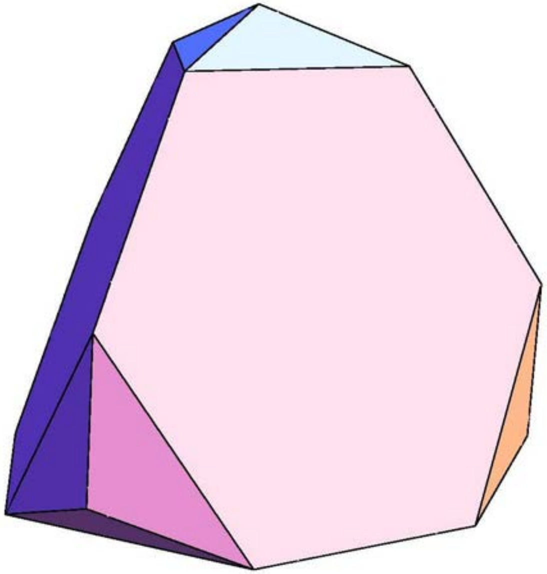


Could you verify that the development in the figure represents the Voronoi domain of the diamond lattice ?

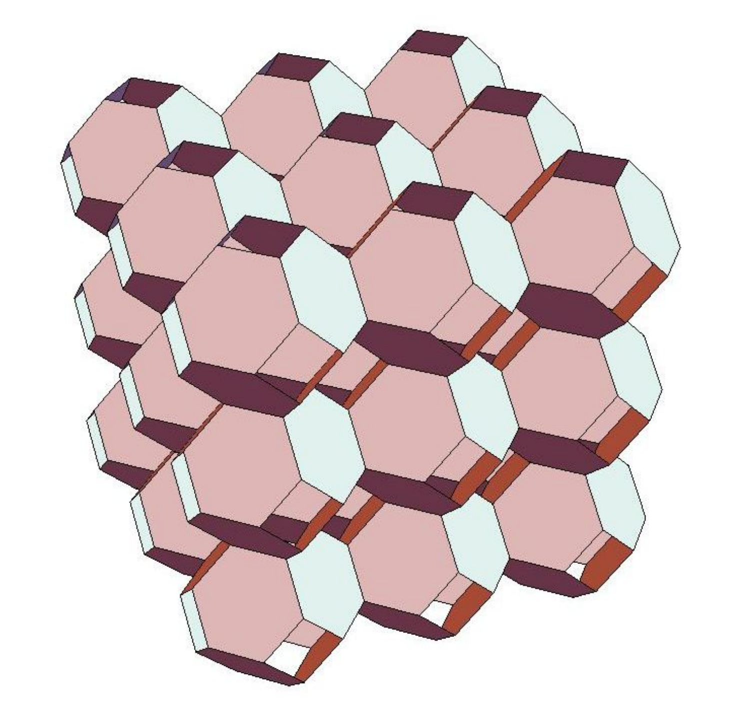
The isotropy group of each vertex of the lattice is isomorphic to the group of isometry of regular tetrahedron which corresponds to $T(2,3,3)$.

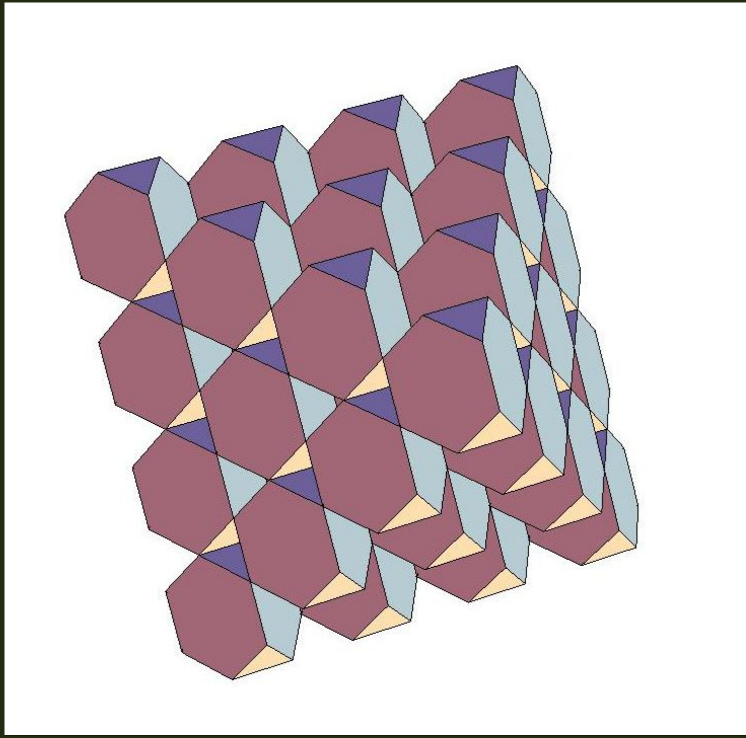


The Voronoi domain of the diamond lattice is obtained from the regular tetrahedron by truncating the tetrahedra of one-third size at the vertices and put the pieces obtained from one deleted tetrahedron by dividing from the barycenter.

Do you understand the way how the space is filled ?

Wikimedia Commons https://commons.wikimedia.org/wiki/File:Georgy_Voronoy.jpg (ref. 2017/5/18)





The hexagons in the boundaries of the Dirichelet domains of the body-centered cubic lattice or those in the boundaries of Voronoi domains of the diamond lattice form surfaces of infinite genus periodically embedded in the space.

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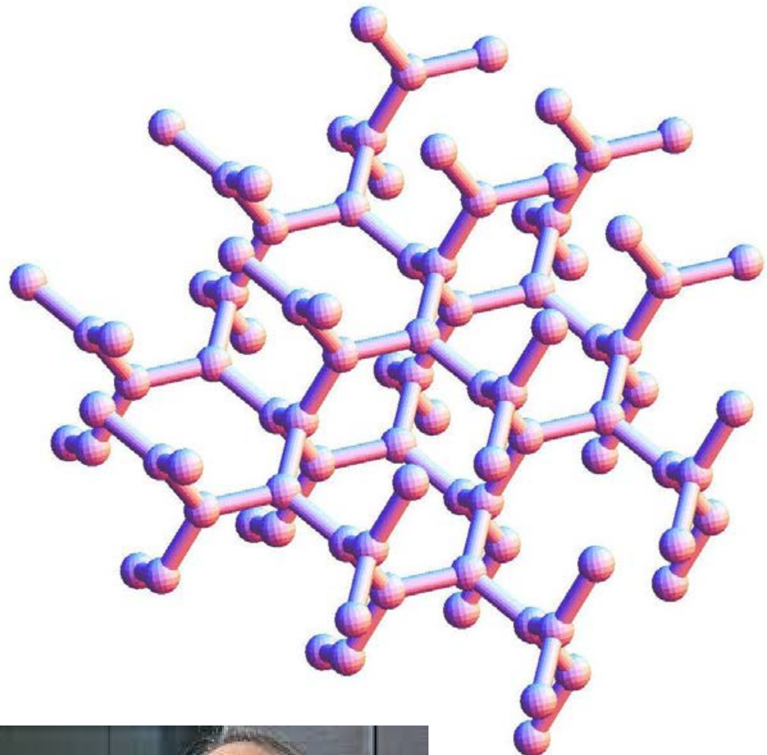
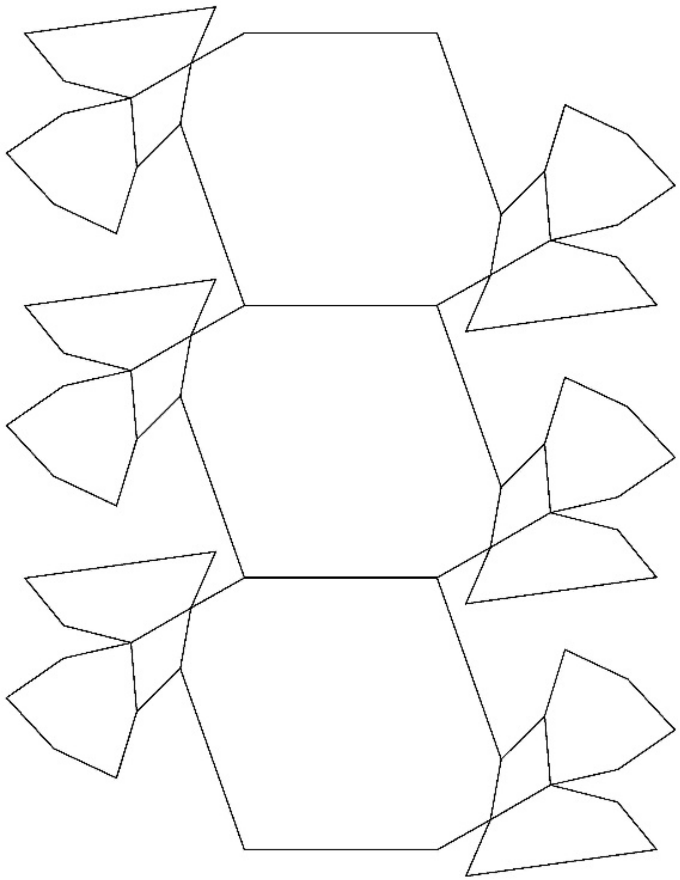


The K4 lattice of Toshikazu Sunada is shown in the figure.

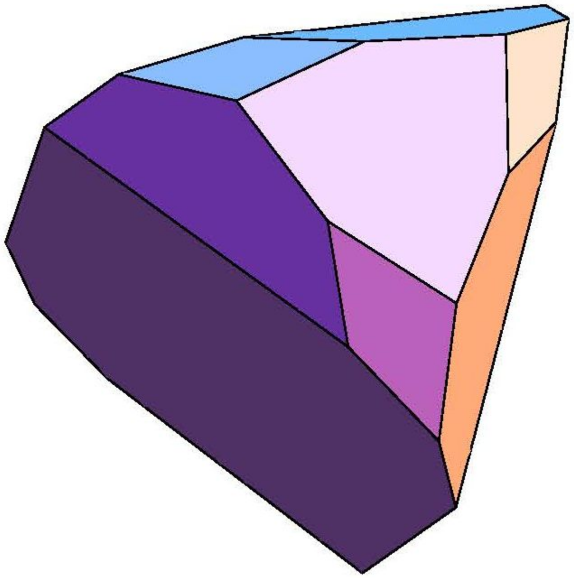
The configuration at each vertex is given as edges joining the vertices of regular triangle and its barycenter. The face angle θ of the planes of the endpoints of edges satisfies $\cos\theta = \frac{1}{3}$.

K4 is the complete graph with 4 vertices and the graph of K4 lattice is the universal abelian covering of K4.



The development of the Voronoi domain of the K4 lattice of Toshikazu Sunada is shown in the figure.

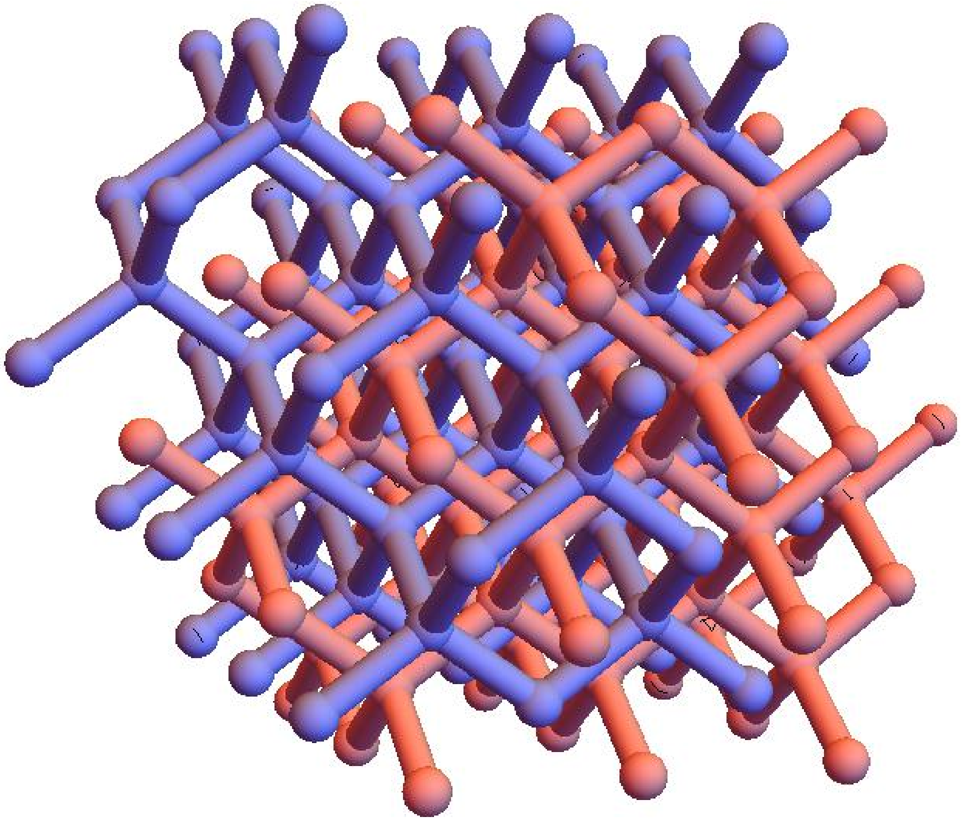
The isotropy group of each vertex of the lattice is isomorphic to D_6.



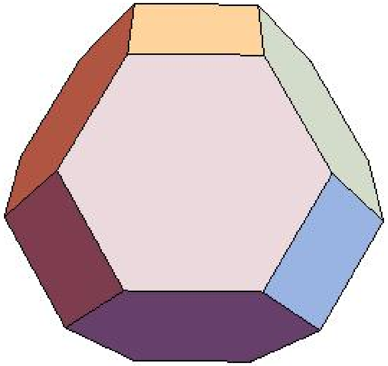
The Voronoi domain of Toshikazu Sunada has 17 faces which looks carved from the triangular cylinder.

This Voronoi domain is different from its mirror image by orientation-preserving isometry.

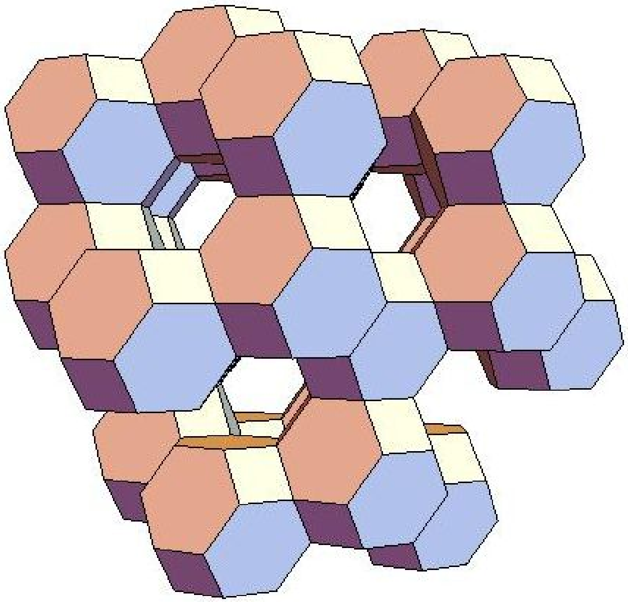
Could you see how the space is filled ?

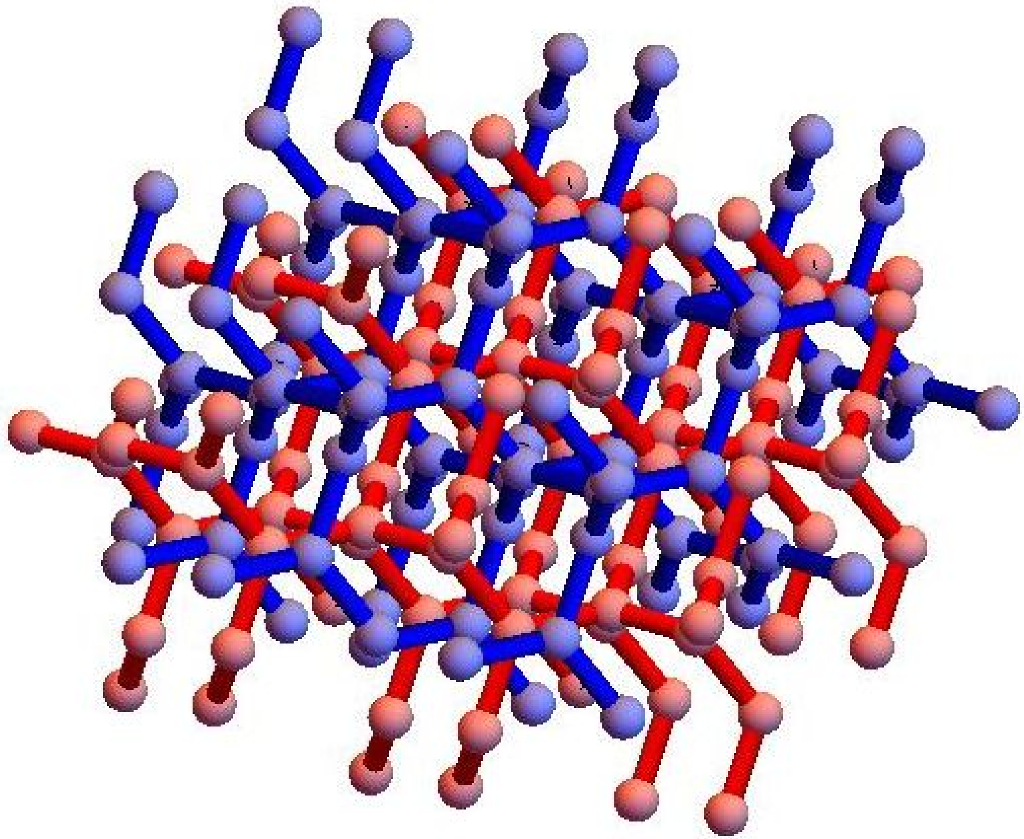


The point reflection of the diamond lattice with respect to a certain point is a core of the complement of the diamond lattice, and these two lattices form the double diamond lattice.



The Voronoi domain of the double diamond lattice is the truncated octahedron.





The point reflection of the K4 lattice with respect to a certain point is the mirror image of the K4 lattice and is a core of the K4 lattice, and these lattices form the double K4 lattice.

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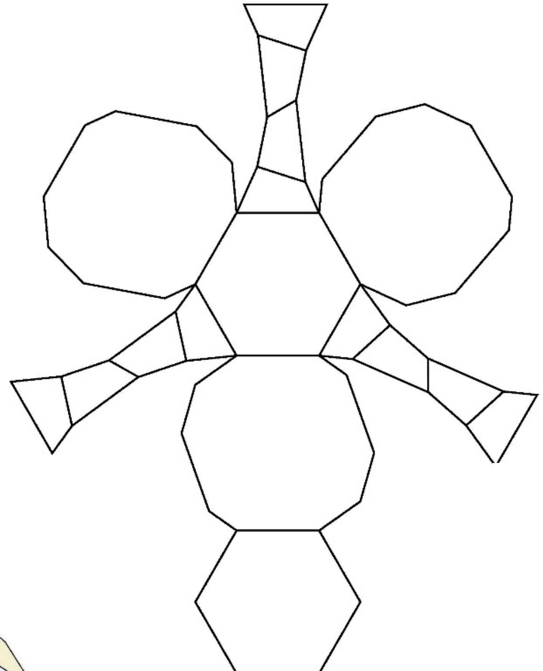
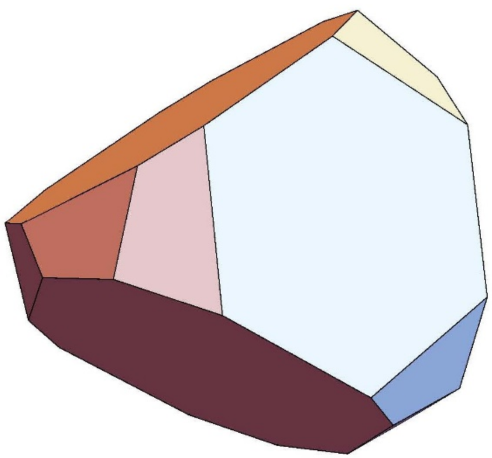
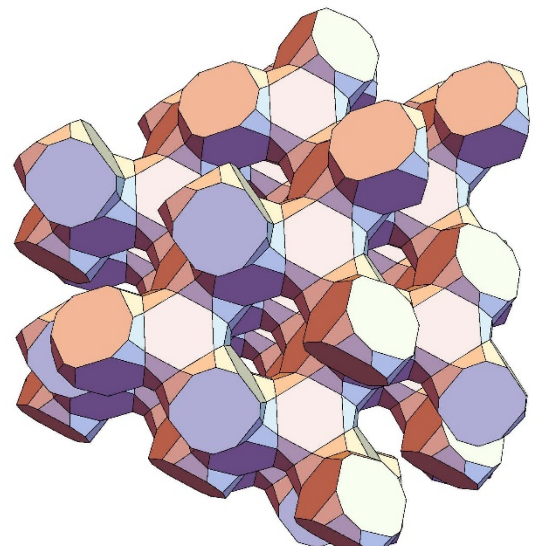
The Voronoi domain of the double K4 lattice is the polyhedron in the figure below.



The cubic lattice and its point reflection form a double lattice. It is very interesting to know whether there other such double lattices.





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