Nonlinear
Finite Element Method

20/12/2004

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Finite Element Method

- Lectures include discussion of the nonlinear finite element method.
- It is preferable to have completed “Introduction to Nonlinear Finite Element Analysis” available in summer session.
- If not, students are required to study on their own before participating this course. Reference:Toshiaki.,Kubo. “Introduction: Tensor Analysis For Nonlinear Finite Element Method” (Hisennkei Yugen Yoso no tameno Tensor Kaiseki no Kiso),Maruzen.
- Lecture references are available and downloadable at http://www.sml.k.u-tokyo.ac.jp/members/nabe/lecture2004 They should be posted on the website by the day before scheduled meeting, and each students are expected to come in with a copy of the reference.
- Lecture notes from previous year are available and downloadable, also at http://www.sml.k.u.tokyo.ac.jp/members/nabe/lecture2003 You may find the course title, ”Advanced Finite Element Method” but the contents covered are the same I will cover this year.
- I will assign the exercises from this year, and expect the students to hand them in during the following lecture. They are not the requirements and they will not be graded, however it is important to actually practice calculate in deeper understanding the finite element method.
- For any questions, contact me at nabe@sml.k.u-tokyo.ac.jp

Nonlinear Finite Element Method Lecture Schedule

1. 10/ 4 Finite element analysis in boundary value problems and the differential equations
2. 10/18 Finite element analysis in linear elastic body
3. 10/25 Isoparametric solid element (program)
4. 11/ 1 Numerical solution and boundary condition processing for system of linear equations（with exercises）
5. 11/ 8 Basic program structure of the linear finite element method(program)
6. 11/15 Finite element formulation in geometric nonlinear problems(program)
7. 11/22 Static analysis technique、hyperelastic body and elastic-plastic material for nonlinear equations (program)
8. 11/29 Exercises for Lecture7
9. 12/ 6 Dynamic analysis technique and eigenvalue analysis in the nonlinear equations
10. 12/13 Structural element
11. 12/20 Numerical solution— skyline method、iterative method for the system of linear equations
12. 1/17 ALE finite element fluid analysis
13. 1/24 ALE finite element fluid analysis

2 Basics to **skyline** method

The characteristics of stiffness matrix in finite element method are,

- Involves higher orders.
- Contains many 0 components (sparse)
- Non-zero components can be found near by diagonal terms(band form)
- Symmetry in normal structure analysis problems
- Non-zero components exist for the anti-symmetric stiffness matrix, also at the location about the diagonal terms.

Skyline method is the technique to take advantage of the characteristics of these facts in addition to some improvement of the Gauss elimination.

For example, suppose the following stiffness matrix is given

$$\begin{bmatrix} K_{11} & K_{12} & 0 & 0 & K_{15} & 0 & 0 & K_{18} \\ K_{21} & K_{22} & K_{23} & K_{24} & 0 & 0 & 0 & K_{28} \\ 0 & K_{32} & K_{33} & K_{34} & 0 & 0 & K_{37} & 0 \\ 0 & K_{42} & K_{43} & K_{44} & 0 & K_{46} & K_{47} & 0 \\ K_{51} & 0 & 0 & 0 & K_{55} & K_{56} & K_{57} & 0 \\ 0 & 0 & 0 & K_{64} & K_{65} & K_{66} & K_{67} & 0 \\ 0 & 0 & K_{73} & K_{74} & K_{75} & K_{76} & K_{77} & K_{78} \\ K_{81} & K_{82} & 0 & 0 & 0 & 0 & K_{87} & K_{88} \end{bmatrix} \tag{1}$$

For each row or a column, a list of the numbers of the smallest row or a column including the non-zero components to be $m_j \quad (j = 1 \ldots n)$. For example, equation (1) may have $m_j = \{1, 1, 2, 2, 1, 4, 3, 1\}$.

This m_j stays the same for $[L]$ and $[U]$ given by the factorization of $[K]$

Apparently, equation (1) can be expressed as the following by triangular factorization.

$$\left[K^{(1)}\right] = \begin{bmatrix} K_{11}^{(1)} & K_{12}^{(1)} & 0 & 0 & K_{15}^{(1)} & 0 & 0 & K_{18}^{(1)} \\ K_{21}^{(1)} & K_{22}^{(1)} & K_{23}^{(1)} & K_{24}^{(1)} & 0 & 0 & 0 & K_{28}^{(1)} \\ 0 & K_{32}^{(1)} & K_{33}^{(1)} & K_{34}^{(1)} & 0 & 0 & K_{37}^{(1)} & 0 \\ 0 & K_{42}^{(1)} & K_{43}^{(1)} & K_{44}^{(1)} & 0 & K_{46}^{(1)} & K_{47}^{(1)} & 0 \\ K_{51}^{(1)} & 0 & 0 & 0 & K_{55}^{(1)} & K_{56}^{(1)} & K_{57}^{(1)} & 0 \\ 0 & 0 & 0 & K_{64}^{(1)} & K_{65}^{(1)} & K_{66}^{(1)} & K_{67}^{(1)} & 0 \\ 0 & 0 & K_{73}^{(1)} & K_{74}^{(1)} & K_{75}^{(1)} & K_{76}^{(1)} & K_{77}^{(1)} & K_{78}^{(1)} \\ K_{81}^{(1)} & K_{82}^{(1)} & 0 & 0 & 0 & 0 & K_{87}^{(1)} & K_{88}^{(1)} \end{bmatrix} \tag{2}$$

$$[L_1^{-1}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -L_{21} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -L_{51} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -L_{81} & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$\left[K^{(2)}\right] = \left[L_1^{-1}\right]\left[K^{(1)}\right] = \begin{bmatrix} K_{11}^{(2)} & K_{12}^{(2)} & 0 & 0 & K_{15}^{(2)} & 0 & 0 & K_{18}^{(2)} \\ 0 & K_{22}^{(2)} & K_{23}^{(2)} & K_{24}^{(2)} & K_{25}^{(2)} & 0 & 0 & K_{28}^{(2)} \\ 0 & K_{32}^{(2)} & K_{33}^{(2)} & K_{34}^{(2)} & 0 & 0 & K_{37}^{(2)} & 0 \\ 0 & K_{42}^{(2)} & K_{43}^{(2)} & K_{44}^{(2)} & 0 & K_{46}^{(2)} & K_{47}^{(2)} & 0 \\ 0 & K_{52}^{(2)} & 0 & 0 & K_{55}^{(2)} & K_{56}^{(2)} & K_{57}^{(2)} & 0 \\ 0 & 0 & 0 & K_{64}^{(2)} & K_{65}^{(2)} & K_{66}^{(2)} & K_{67}^{(2)} & 0 \\ 0 & 0 & K_{73}^{(2)} & K_{74}^{(2)} & K_{75}^{(2)} & K_{76}^{(2)} & K_{77}^{(2)} & K_{78}^{(2)} \\ 0 & K_{82}^{(2)} & 0 & 0 & K_{85}^{(2)} & 0 & K_{87}^{(2)} & K_{88}^{(2)} \end{bmatrix} \tag{4}$$

$$\left[L_2^{-1}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -L_{32} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -L_{42} & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -L_{52} & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -L_{82} & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

$$\left[K^{(3)}\right] = \left[L_2^{-1}\right]\left[K^{(2)}\right] = \begin{bmatrix} K_{11}^{(3)} & K_{12}^{(3)} & 0 & 0 & K_{15}^{(3)} & 0 & 0 & K_{18}^{(3)} \\ 0 & K_{22}^{(3)} & K_{23}^{(3)} & K_{24}^{(3)} & K_{25}^{(3)} & 0 & 0 & K_{28}^{(3)} \\ 0 & 0 & K_{33}^{(3)} & K_{34}^{(3)} & K_{35}^{(3)} & 0 & K_{37}^{(3)} & K_{38}^{(3)} \\ 0 & 0 & K_{43}^{(3)} & K_{44}^{(3)} & K_{45}^{(3)} & K_{46}^{(3)} & K_{47}^{(3)} & K_{48}^{(3)} \\ 0 & 0 & K_{53}^{(3)} & K_{54}^{(3)} & K_{55}^{(3)} & K_{56}^{(3)} & K_{57}^{(3)} & K_{58}^{(3)} \\ 0 & 0 & 0 & K_{64}^{(3)} & K_{65}^{(3)} & K_{66}^{(3)} & K_{67}^{(3)} & 0 \\ 0 & 0 & K_{73}^{(3)} & K_{74}^{(3)} & K_{75}^{(3)} & K_{76}^{(3)} & K_{77}^{(3)} & K_{78}^{(3)} \\ 0 & 0 & K_{83}^{(3)} & K_{84}^{(3)} & K_{85}^{(3)} & 0 & K_{87}^{(3)} & K_{88}^{(3)} \end{bmatrix} \tag{6}$$

$$\left[L_3^{-1}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -L_{43} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -L_{53} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -L_{73} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -L_{83} & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

$$\left[K^{(4)}\right] = \left[L_3^{-1}\right]\left[K^{(3)}\right] = \begin{bmatrix} K_{11}^{(4)} & K_{12}^{(4)} & 0 & 0 & K_{15}^{(4)} & 0 & 0 & K_{18}^{(4)} \\ 0 & K_{22}^{(4)} & K_{23}^{(4)} & K_{24}^{(4)} & K_{25}^{(4)} & 0 & 0 & K_{28}^{(4)} \\ 0 & 0 & K_{33}^{(4)} & K_{34}^{(4)} & K_{35}^{(4)} & 0 & K_{37}^{(4)} & K_{38}^{(4)} \\ 0 & 0 & 0 & K_{44}^{(4)} & K_{45}^{(4)} & K_{46}^{(4)} & K_{47}^{(4)} & K_{48}^{(4)} \\ 0 & 0 & 0 & K_{54}^{(4)} & K_{55}^{(4)} & K_{56}^{(4)} & K_{57}^{(4)} & K_{58}^{(4)} \\ 0 & 0 & 0 & K_{64}^{(4)} & K_{65}^{(4)} & K_{66}^{(4)} & K_{67}^{(4)} & 0 \\ 0 & 0 & 0 & K_{74}^{(4)} & K_{75}^{(4)} & K_{76}^{(4)} & K_{77}^{(4)} & K_{78}^{(4)} \\ 0 & 0 & 0 & K_{84}^{(4)} & K_{85}^{(4)} & 0 & K_{87}^{(4)} & K_{88}^{(4)} \end{bmatrix} \tag{8}$$

$$\left[L_4^{-1}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -L_{54} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -L_{64} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -L_{74} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -L_{84} & 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

$$
\left[K^{(5)}\right] = \left[L_4^{-1}\right]\left[K^{(4)}\right] = \begin{bmatrix}
K_{11}^{(5)} & K_{12}^{(5)} & 0 & 0 & K_{15}^{(5)} & 0 & 0 & K_{18}^{(5)} \\
0 & K_{22}^{(5)} & K_{23}^{(5)} & K_{24}^{(5)} & K_{25}^{(5)} & 0 & 0 & K_{28}^{(5)} \\
0 & 0 & K_{33}^{(5)} & K_{34}^{(5)} & K_{35}^{(5)} & 0 & K_{37}^{(5)} & K_{38}^{(5)} \\
0 & 0 & 0 & K_{44}^{(5)} & K_{45}^{(5)} & K_{46}^{(5)} & K_{47}^{(5)} & K_{48}^{(5)} \\
0 & 0 & 0 & 0 & K_{55}^{(5)} & K_{56}^{(5)} & K_{57}^{(5)} & K_{58}^{(5)} \\
0 & 0 & 0 & 0 & K_{65}^{(5)} & K_{66}^{(5)} & K_{67}^{(5)} & K_{68}^{(5)} \\
0 & 0 & 0 & 0 & K_{75}^{(5)} & K_{76}^{(5)} & K_{77}^{(5)} & K_{78}^{(5)} \\
0 & 0 & 0 & 0 & K_{85}^{(5)} & K_{86}^{(5)} & K_{87}^{(5)} & K_{88}^{(5)}
\end{bmatrix} \tag{10}
$$

$$
\left[L_5^{-1}\right] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -L_{65} & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -L_{75} & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -L_{85} & 0 & 0 & 1
\end{bmatrix} \tag{11}
$$

$$\left[K^{(6)}\right] = \left[L_5^{-1}\right]\left[K^{(5)}\right] = \begin{bmatrix} K_{11}^{(6)} & K_{12}^{(6)} & 0 & 0 & K_{15}^{(6)} & 0 & 0 & K_{18}^{(6)} \\ 0 & K_{22}^{(6)} & K_{23}^{(6)} & K_{24}^{(6)} & K_{25}^{(6)} & 0 & 0 & K_{28}^{(6)} \\ 0 & 0 & K_{33}^{(6)} & K_{34}^{(6)} & K_{35}^{(6)} & 0 & K_{37}^{(6)} & K_{38}^{(6)} \\ 0 & 0 & 0 & K_{44}^{(6)} & K_{45}^{(6)} & K_{46}^{(6)} & K_{47}^{(6)} & K_{48}^{(6)} \\ 0 & 0 & 0 & 0 & K_{55}^{(6)} & K_{56}^{(6)} & K_{57}^{(6)} & K_{58}^{(6)} \\ 0 & 0 & 0 & 0 & 0 & K_{66}^{(6)} & K_{67}^{(6)} & K_{68}^{(6)} \\ 0 & 0 & 0 & 0 & 0 & K_{76}^{(6)} & K_{77}^{(6)} & K_{78}^{(6)} \\ 0 & 0 & 0 & 0 & 0 & K_{86}^{(6)} & K_{87}^{(6)} & K_{88}^{(6)} \end{bmatrix} \tag{12}$$

$$\left[L_6^{-1}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -L_{76} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -L_{86} & 0 & 1 \end{bmatrix} \tag{13}$$

$$\left[K^{(7)}\right] = \left[L_6^{-1}\right]\left[K^{(6)}\right] = \begin{bmatrix} K_{11}^{(7)} & K_{12}^{(7)} & 0 & 0 & K_{15}^{(7)} & 0 & 0 & K_{18}^{(7)} \\ 0 & K_{22}^{(7)} & K_{23}^{(7)} & K_{24}^{(7)} & K_{25}^{(7)} & 0 & 0 & K_{28}^{(7)} \\ 0 & 0 & K_{33}^{(7)} & K_{34}^{(7)} & K_{35}^{(7)} & 0 & K_{37}^{(7)} & K_{38}^{(7)} \\ 0 & 0 & 0 & K_{44}^{(7)} & K_{45}^{(7)} & K_{46}^{(7)} & K_{47}^{(7)} & K_{48}^{(7)} \\ 0 & 0 & 0 & 0 & K_{55}^{(7)} & K_{56}^{(7)} & K_{57}^{(7)} & K_{58}^{(7)} \\ 0 & 0 & 0 & 0 & 0 & K_{66}^{(7)} & K_{67}^{(7)} & K_{68}^{(7)} \\ 0 & 0 & 0 & 0 & 0 & 0 & K_{77}^{(7)} & K_{78}^{(7)} \\ 0 & 0 & 0 & 0 & 0 & 0 & K_{87}^{(7)} & K_{88}^{(7)} \end{bmatrix} \tag{14}$$

$$\left[L_7^{-1}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -L_{87} & 1 \end{bmatrix} \tag{15}$$

$$\left[K^{(8)}\right] = [L_7^{-1}]\left[K^{(7)}\right] = \begin{bmatrix} K_{11}^{(8)} & K_{12}^{(8)} & 0 & 0 & K_{15}^{(8)} & 0 & 0 & K_{18}^{(8)} \\ 0 & K_{22}^{(8)} & K_{23}^{(8)} & K_{24}^{(8)} & K_{25}^{(8)} & 0 & 0 & K_{28}^{(8)} \\ 0 & 0 & K_{33}^{(8)} & K_{34}^{(8)} & K_{35}^{(8)} & 0 & K_{37}^{(8)} & K_{38}^{(8)} \\ 0 & 0 & 0 & K_{44}^{(8)} & K_{45}^{(8)} & K_{46}^{(8)} & K_{47}^{(8)} & K_{48}^{(8)} \\ 0 & 0 & 0 & 0 & K_{55}^{(8)} & K_{56}^{(8)} & K_{57}^{(8)} & K_{58}^{(8)} \\ 0 & 0 & 0 & 0 & 0 & K_{66}^{(8)} & K_{67}^{(8)} & K_{68}^{(8)} \\ 0 & 0 & 0 & 0 & 0 & 0 & K_{77}^{(8)} & K_{78}^{(8)} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K_{88}^{(8)} \end{bmatrix} \tag{16}$$

Hence [*L*] can be written by,

$$[L] = [L_1]\,[L_2]\ldots[L_7] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_{32} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_{42} & L_{43} & 1 & 0 & 0 & 0 & 0 \\ L_{51} & L_{52} & L_{53} & L_{54} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_{64} & L_{65} & 1 & 0 & 0 \\ 0 & 0 & L_{73} & L_{74} & L_{75} & L_{76} & 1 & 0 \\ L_{81} & L_{82} & L_{83} & L_{84} & L_{85} & L_{86} & L_{87} & 1 \end{bmatrix} \tag{17}$$

Considering the fact, $[K^{(8)}] = [D][U]$, for $[L]$ and $[U]$, we can observe m_j = 1,1,2,2,1,4,3 and m_j in initial$[K]$ are the identical. Nevertheless, the components that are closer to the diagonal terms than that of *mj,* such as $K_{25}, K_{35}, K_{45}, K_{38}, K_{48}, K_{58}, K_{68}$ the values take non-zero in general by the triangular factorization.

Using these characteristics, Gauss elimination can be rationalized. $[L]$,and$[U]$ can be expressed by,

$$[L] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ L_{21} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_{32} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_{42} & L_{43} & 1 & 0 & 0 & 0 & 0 \\ L_{51} & L_{52} & L_{53} & L_{54} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_{64} & L_{65} & 1 & 0 & 0 \\ 0 & 0 & L_{73} & L_{74} & L_{75} & L_{76} & 1 & 0 \\ L_{81} & L_{82} & L_{83} & L_{84} & L_{85} & L_{86} & L_{87} & 1 \end{bmatrix} \quad (18)$$

$$[U] = \begin{bmatrix} 1 & U_{12} & 0 & 0 & U_{15} & 0 & 0 & U_{18} \\ 0 & 1 & U_{23} & U_{24} & U_{25} & 0 & 0 & U_{28} \\ 0 & 0 & 1 & U_{34} & U_{35} & 0 & U_{37} & U_{38} \\ 0 & 0 & 0 & 1 & U_{45} & U_{46} & U_{47} & U_{48} \\ 0 & 0 & 0 & 0 & 1 & U_{56} & U_{57} & U_{58} \\ 0 & 0 & 0 & 0 & 0 & 1 & U_{67} & U_{68} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & U_{78} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (19)$$

Using $[L]$ and $[U]$, $[K] = [L][D][U]$ can be re-calculated as, Row 1,2, and 3

$$\begin{bmatrix} D_{11} & D_{11}U_{12} & 0 \\ L_{21}D_{11} & L_{21}D_{11}U_{12} + D_{22} & D_{22}U_{23} \\ 0 & L_{32}D_{22} & L_{32}D_{22}U_{23} + D_{33} \\ 0 & L_{42}D_{22} & L_{42}D_{22}U_{23} + L_{43}D_{33} \\ L_{51}D_{11} & L_{51}D_{11}U_{12} + L_{52}D_{22} & L_{52}D_{22}U_{23} + L_{53}D_{33} \\ 0 & 0 & 0 \\ 0 & 0 & L_{73}D_{33} \\ L_{81}D_{11} & L_{81}D_{11}U_{12} + L_{82}D_{22} & L_{82}D_{22}U_{23} + L_{83}D_{33} \end{bmatrix} \quad (20)$$

Row 4 and 5

$$\begin{bmatrix} D_{11}U_{15} & 0 \\ L_{21}D_{11}U_{15} + D_{22}U_{25} & 0 \\ L_{32}D_{22}U_{25} + D_{33}U_{35} & 0 \\ L_{42}D_{22}U_{25} + L_{43}D_{33}U_{35} + D_{44}U_{45} & D_{44}U_{46} \\ L_{51}D_{11}U_{15} + L_{52}D_{22}U_{25} + L_{53}D_{33}U_{35} + L_{54}D_{44}U_{45} + D_{55} & L_{54}D_{44}U_{46} + D_{55}U_{56} \\ L_{64}D_{44}U_{45} + L_{65}D_{55} & L_{64}D_{44}U_{46} + L_{65}D_{55}U_{56} + D_{66} \\ L_{73}D_{33}U_{35} + L_{74}D_{44}U_{45} + L_{75}D_{55} & L_{74}D_{44}U_{46} + L_{75}D_{55}U_{56} + L_{76}D_{66} \\ L_{81}D_{11}U_{15} + L_{82}D_{22}U_{25} + L_{83}D_{33}U_{35} + L_{84}D_{44}U_{45} + L_{85}D_{55} & L_{84}D_{44}U_{46} + L_{85}D_{55}U_{56} + L_{86}D_{66} \end{bmatrix} \quad (21)$$

Row 7

$$\begin{bmatrix} 0 \\ 0 \\ D_{33}U_{37} \\ L_{43}D_{33}U_{27} + D_{44}U_{47} \\ L_{53}D_{33}U_{37} + L_{54}D_{44}U_{47} + D_{55}U_{57} \\ L_{64}D_{44}U_{47} + L_{65}D_{55}U_{57} + D_{66}U_{67} \\ L_{73}D_{33}U_{35} + L_{74}D_{44}U_{45} + L_{75}D_{55}U_{57} + L_{76}D_{66}U_{67} + D_{77} \\ L_{83}D_{33}U_{37} + L_{84}D_{44}U_{47} + L_{85}D_{55}U_{57} + L_{86}D_{66}U_{67} + L_{87}D_{77} \end{bmatrix} \tag{22}$$

Row 8

$$\begin{bmatrix} D_{11}U_{18} \\ L_{21}D_{11}U_{18} + D_{22}U_{28} \\ L_{32}D_{22}U_{28} + D_{33}U_{38} \\ L_{42}D_{22}U_{28} + L_{43}D_{33}U_{38} + D_{44}U_{48} \\ L_{51}D_{11}U_{18} + L_{52}D_{22}U_{28} + L_{53}D_{33}U_{38} + L_{52}D_{44}U_{48} + D_{55}U_{58} \\ L_{64}D_{44}U_{48} + L_{65}D_{55}U_{58} + D_{66}U_{68} \\ L_{73}D_{33}U_{38} + L_{74}D_{44}U_{48} + L_{75}D_{55}U_{58} + L_{76}D_{66}U_{68} + D_{77}U_{78} \\ L_{81}D_{11}U_{18} + L_{82}D_{22}U_{28} + L_{83}D_{33}U_{38} + L_{84}D_{44}U_{48} + L_{85}D_{55}U_{58} + L_{86}D_{66}U_{68} + L_{87}D_{77}U_{78} + D_{88} \end{bmatrix} \tag{23}$$

Generally, this can be expressed as in the following with using the index.

$$\left\{\begin{array}{ll} K_{ij} = D_{ii}U_{ij} & \text{上三角} \\ K_{ji} = L_{ji}D_{ii} & \text{下三角} \end{array}\right\} \quad i = m_j$$

$$\left\{\begin{array}{ll} K_{ij} = \displaystyle\sum_{k=\max(m_i,m_j)}^{i-1} L_{ik}D_{kk}U_{kj} + D_{ii}U_{ij} & \text{上三角} \\ K_{ji} = \displaystyle\sum_{k=\max(m_i,m_j)}^{i-1} L_{jk}D_{kk}U_{ki} + L_{ji}D_{ii} & \text{下三角} \end{array}\right\} \quad i = m_j + 1 \ldots j - 1$$

$$K_{jj} = \sum_{k=m_j}^{j-1} L_{jk}D_{kk}U_{kj} + D_{jj} \quad \text{対角項}$$

From this relational expression, $[L]$, $[D]$ and $[U]$ can be calculated as following in turn.

$$\textbf{for} \quad j = 1$$

$$D_{11} = K_{11}$$

$$\textbf{for} \quad j = 2$$

$$U_{12} = K_{12}/D_{11}$$

$$L_{21} = K_{21}/D_{11}$$

$$D_{22} = K_{22} - L_{21}D_{11}U_{12}$$

```
for  j = 3 ~ n
    for  i = m_j + 1 ~ j − 1
        for  k = max(m_i, m_j) ~ i − 1
            temp_u = temp_u + L_ik{D_kk U_kj}
            temp_l = temp_l + {L_jk D_kk}U_ki
        end for
        (D_ii U_ij) = K_ij − temp_u
        (L_ji D_ii) = K_ji − temp_l
    end for
    for  i = m_j ~ j − 1
        U_ij = (D_ii U_ij)/D_ii
        L_ji = (L_ji D_ii)/D_ii
    end for
    for  k = m_j ~ j − 1
        temp = temp + L_jk D_kk U_kj
    end for
    D_jj = K_jj − temp
end for
```