Nonlinear Finite Element Method

17/01/2005

Nonlinear
Finite Element Method

- Lectures include discussion of the nonlinear finite element method.
- It is preferable to have completed “Introduction to Nonlinear Finite Element Analysis” available in summer session.
- If not, students are required to study on their own before participating this course. Reference:Toshiaki.,Kubo. “Introduction: Tensor Analysis For Nonlinear Finite Element Method” (Hisennkei Yugen Yoso no tameno Tensor Kaiseki no Kiso),Maruzen.
- Lecture references are available and downloadable at http://www.sml.k.u-tokyo.ac.jp/members/nabe/lecture2004 They should be posted on the website by the day before scheduled meeting, and each students are expected to come in with a copy of the reference.
- Lecture notes from previous year are available and downloadable, also at http://www.sml.k.u.tokyo.ac.jp/members/nabe/lecture2003 You may find the course title, ”Advanced Finite Element Method” but the contents covered are the same I will cover this year.
- I will assign the exercises from this year, and expect the students to hand them in during the following lecture. They are not the requirements and they will not be graded, however it is important to actually practice calculate in deeper understanding the finite element method.
- For any questions, contact me at nabe@sml.k.u-tokyo.ac.jp

Nonlinear Finite Element Method Lecture Schedule

1. 10/ 4 Finite element analysis in boundary value problems and the differential equations
2. 10/18 Finite element analysis in linear elastic body
3. 10/25 Isoparametric solid element (program)
4. 11/ 1 Numerical solution and boundary condition processing for system of linear equations（with exercises）
5. 11/ 8 Basic program structure of the linear finite element method(program)
6. 11/15 Finite element formulation in geometric nonlinear problems(program)
7. 11/22 Static analysis technique、hyperelastic body and elastic-plastic material for nonlinear equations (program)
8. 11/29 Exercises for Lecture7
9. 12/ 6 Dynamic analysis technique and eigenvalue analysis in the nonlinear equations
10. 12/13 Structural element
11. 12/20 Numerical solution— skyline method、iterative method for the system of linear equations
12. 1/17 ALE finite element fluid analysis
13. 1/24 ALE finite element fluid analysis

Fundamental Equation for ALE Method in Fluid Analysis

The coordinates system expressed in Lagrange notation is called physical coordinates system(Lagrange coordinates system), while the coordinates expressed in Euler notation is called spatial coordinates system(Euler coordinates). These coordinates systems are set by the arbitrary independent coordinates system as indicated in Fig.2.1, and the way to express the movement of a substance in such coordinates is called reference representation (ALE representation).

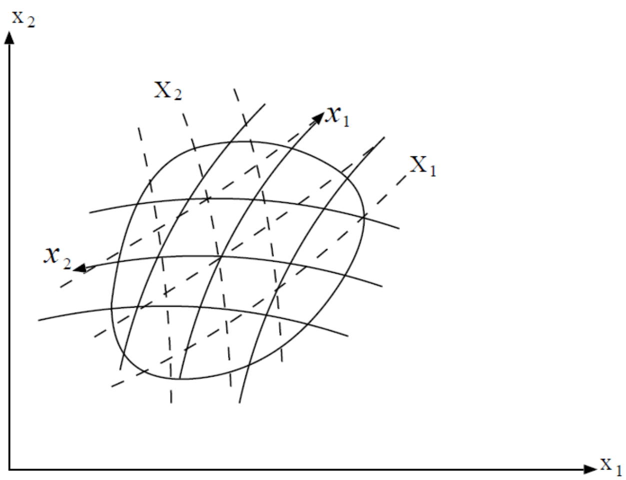

図 2.1: Lagrange 座標系 $\boldsymbol{X}$，Euler 座標系 $\boldsymbol{x}$，及び ALE 座標系 $\boldsymbol{\chi}$ の概念図

Now, let us express a point in the region of analysis by Lagrange, Euler and ALE representation.

$$\begin{cases} \text{Lagrange 表示} & : \quad R_X \quad , \quad \boldsymbol{X} \\ \text{Euler 表示} & : \quad R_x \quad , \quad \boldsymbol{x} \\ \text{ALE 表示} & : \quad R_\chi \quad , \quad \boldsymbol{\chi} \end{cases} \tag{2.1}$$

Consider now an arbitrary quantity of *f* , which expresses the fields such as scalar ϕ, vector$\boldsymbol{b}$ and tensor$\boldsymbol{A}$ to define their Lagrange, Euler , and ALE representations as $f(\boldsymbol{X},t)$, $f(\boldsymbol{X}(\boldsymbol{x},t),t)$and $f(\boldsymbol{X}(\boldsymbol{\chi},t),t)$ then further consider their various time derivative functions to obtain the following transformations.

1.Relational expression of the actual time derivative function and substance time derivative function

$$
\begin{aligned}
\frac{d\,f}{d\,t} &= \underbrace{\left.\frac{\partial f(\boldsymbol{X},t)}{\partial t}\right|_{\boldsymbol{X}}}_{\text{物質時間導関数}} + \frac{\partial f(\boldsymbol{X},t)}{\partial X_i}\underbrace{\left.\frac{dX_i}{dt}\right|_{\boldsymbol{X}}}_{=0} \\
&= \left.\frac{\partial f(\boldsymbol{X},t)}{\partial t}\right|_{\boldsymbol{X}} \qquad (2.2)
\end{aligned}
$$

2. Relational expression of the actual time derivative function (substance time derivative function) and spatial time derivative function.

$$
\begin{aligned}
\frac{d\,f}{d\,t} &= \underbrace{\left.\frac{\partial f(\boldsymbol{x},t)}{\partial t}\right|_{\boldsymbol{x}}}_{\text{空間時間導関数}} + \frac{\partial f(\boldsymbol{x},t)}{\partial x_i}\underbrace{\left.\frac{\partial x_i(\boldsymbol{X},t)}{\partial t}\right|_{\boldsymbol{X}}}_{\equiv v_i} \\
&= \left.\frac{\partial f(\boldsymbol{x},t)}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{v}\cdot(\nabla_{x}\otimes f) \qquad (2.3)
\end{aligned}
$$

3. Relational expression of the time derivative function(substance time derivative function) and reference time derivative function.

$$
\begin{aligned}
\frac{d\,f}{d\,t} &= \underbrace{\left.\frac{\partial f(\boldsymbol{\chi},t)}{\partial t}\right|_{\boldsymbol{\chi}}}_{\text{参照時間導関数}} + \frac{\partial f(\boldsymbol{\chi},t)}{\partial \chi_i}\underbrace{\left.\frac{\partial \chi_i(\boldsymbol{X},t)}{\partial t}\right|_{\boldsymbol{X}}}_{\equiv w_i} \\
&= \left.\frac{\partial f(\boldsymbol{\chi},t)}{\partial t}\right|_{\boldsymbol{\chi}} + \boldsymbol{w}\cdot(\nabla_{\chi}\otimes f) \qquad (2.4)
\end{aligned}
$$

Clearly, in the time derivative functions except for the substance time derivative function, we can observe advective derivative in the right hand side the second term. Generally, there may be appearance of advective term and convective term. Since the spatial time derivative function involves a fixed observer in space, and the reference time derivative function involves the time variation observed by an observer traveling arbitrarily, thus it refers to the rate of change in the physical quantity *f* relativistiaclly delivered by the movement of an observer and the substance point.

The physical interpretations of the various velocities found here are stated in the following.

1. Velocity of substance point ***v*** about Euler coordinates system

$$v_i = \left. \frac{\partial x_i(\boldsymbol{X}, t)}{\partial t} \right|_{\boldsymbol{X}} \tag{2.5}$$

2. Velocity of substance point ***w*** about reference coordinates system

$$w_i = \left. \frac{\partial \chi_i(\boldsymbol{X}, t)}{\partial t} \right|_{\boldsymbol{X}} \tag{2.6}$$

3. Velocity of reference coordinates system ***v*** about Euler coordinates system

$$\widehat{v}_i = \left. \frac{\partial x_i(\boldsymbol{\chi}, t)}{\partial t} \right|_{\boldsymbol{\chi}} \tag{2.7}$$

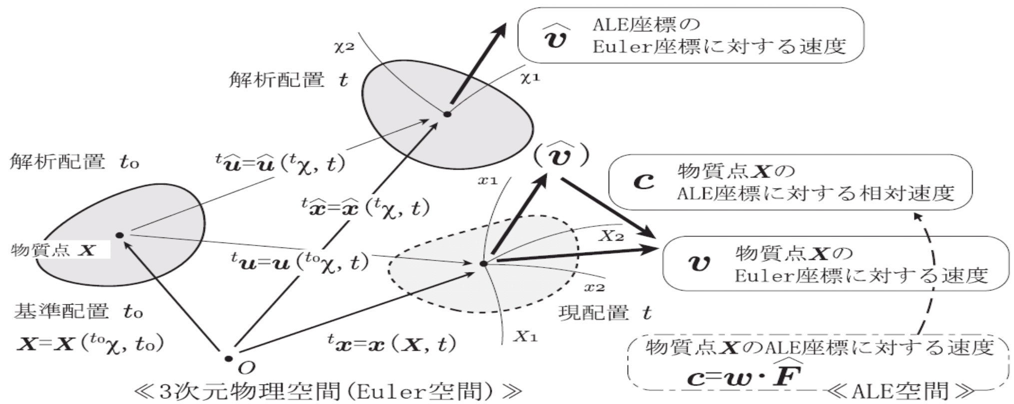

図 2.2: ALE 法における配置と速度関係図

Therefore, the above substance time derivative function $\partial/\partial t|_X$ can be transformed to the reference time derivative function $\partial/\partial t|_\chi$ by using $f \equiv \rho$ in Eq.(2.11) . The ALE conservation of mass in Euler representation can be given by,

$$\left.\frac{\partial \rho}{\partial t}\right|_{\chi} + \boldsymbol{c} \cdot (\nabla_x \rho) + \rho \nabla_x \cdot \boldsymbol{v} = 0 \tag{2.12}$$

The components are expressed by,

$$\left.\frac{\partial \rho}{\partial t}\right|_{\chi} + c_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial v_i}{\partial x_i} = 0 \quad (\text{in } R_x) \tag{2.13}$$

In this study, we deal with the incompressible fluid so, having taken the complete incompressible equation for conservation, $\nabla_x \cdot \boldsymbol{v} = 0$, numerical instability may occurs, hence introduce the the conservation of the infinitesimal compressibility. This is commonly called *Barotropic flow*, and which adopts the following relation (Ray, S. E., Wren, G. P. and Tezduyar, T. E. 1997 [**?**]).

$$p = \frac{B_1}{B_2}\left[\left(\frac{\rho}{\rho_0}\right)^{B_2} - 1\right] \tag{2.14}$$

Where ρ_0 and B_1 represent the volume density and the volume elasticity at standard atmospheric pressure. r B_2 esents a dimensionless invariable.

Next, we derive a fundamental equation to express the advective term by Euler representation. The advective term is given by ALE method: the second term of the reference time derivative function Eq.(2.4) In Eq.(2.4), take an arbitrary physical quantity *f* at present position vector $\boldsymbol{x}$, then gained by the following velocity relational expression.

$$\left.\frac{\partial x_i(\boldsymbol{X},t)}{\partial t}\right|_{\boldsymbol{X}} = \left.\frac{\partial x_i}{\partial t}\right|_{\boldsymbol{\chi}} + w_j \frac{\partial x_i}{\partial \chi_j} \tag{2.8}$$

$$v_i = \widehat{v}_i + w_j \frac{\partial x_i}{\partial \chi_j} \tag{2.9}$$

Accordingly, if we can introduce the relative velocity $\boldsymbol{c} \equiv \boldsymbol{v} - \boldsymbol{v}$ of substance point for the reference coordinates system,then it is possible to expansion the formulation given by ALE method. This relational expression of velocity becomes a fundamental equation in expressing the advective term of ALE method in Euler representation.

$$c_i \equiv v_i - \widehat{v}_i = w_j \frac{\partial x_i}{\partial \chi_j} \tag{2.10}$$

From above, if we substitute the relational expression of velocity(2.10)into the reference time derivative function(2.4), we may obtain an equation that takes the reference time derivative function at Euler region. This is the fundamental equation of time derivative function for ALE method(*arbitrary Lagrangian–Eulerian method*) .

$$\begin{aligned} \left.\frac{\partial f(\boldsymbol{X},t)}{\partial t}\right|_{\boldsymbol{X}} &= \left.\frac{\partial f(\boldsymbol{\chi},t)}{\partial t}\right|_{\boldsymbol{\chi}} + c_i \frac{\partial f}{\partial x_i} \\ &= \left.\frac{\partial f(\boldsymbol{\chi},t)}{\partial t}\right|_{\boldsymbol{\chi}} + \boldsymbol{c} \cdot (\nabla_x \otimes f) \end{aligned} \tag{2.11}$$

Euler representation of ALE conservation of mass

In ALE method, where $\boldsymbol{v} = \boldsymbol{v}$, the coordinates system may be degenerated and given by the Lagrange coordinates system , while where $\boldsymbol{v} = 0$, the system degenerates into the Euler coordinates system. Based on the facts, the Lagrange method and Euler method can be considered as one systematically developed method so, in actual problems, if we can control the mesh velocity v to travel along the displacement boundary plane, we can rationally deal with the displacement boundary problems. Moreover, as Fig.2.2 clearly indicates, the discussion is being constructed by the analytical configuration (ALE mesh), and the actual standard configuration at the given time hence, the present configuration should be left both unknown, therefore, in a case where the constitutive equations are based on the flow velocity with no requirement of information on deformation, such as fluid, we may expect a great effect. When applying into a structure, the standard configuration of ALE mesh is being modified along with the time passing, thus if we can control ALE mesh to avoid the deformation in mesh, we may possibly continue the analysis without conducting re-meshing.

$$\left.\frac{\partial \rho}{\partial t}\right|_{\boldsymbol{X}} + \rho \nabla_x \cdot \boldsymbol{v} = 0 \tag{1.10}$$

$$B = \rho \left.\frac{\partial p}{\partial \rho}\right|_{\theta} \tag{2.15}$$

This Barotropic flow equation(2.14) can be derived from following hypothesis equation of the bulk modulus.

$$B = B_1 + B_2 p \tag{2.16}$$

Assuming the isothermal variation consition(θ=Const), the following equation may be yield by Eq.(2.15).

$$\frac{\partial p}{\partial \rho} = \frac{B}{\rho} \tag{2.17}$$

Furthermore, by ignoring the influence of bulk modulus pressure p, we assume B to be invariable. Using the facts, Euler representation of ALE conservation of mass(2.13) can be re-written as following(Huerta, A. and Liu, W. K. 1988 [**?**]).

$$\frac{1}{B}\left.\frac{\partial p}{\partial t}\right|_{\boldsymbol{\chi}} + \frac{1}{B} c_i \frac{\partial p}{\partial x_i} + \frac{\partial v_i}{\partial x_i} = 0 \qquad (\text{in } R_x) \tag{2.18}$$

The equation above is the infinitesimal compressible ALE conservation of mass(series equation).

Euler representation of ALE Navier–Stokes equation

Now, we consider transforming the Cauchy's law of motion into ALE representation.(Wu, W. Y. [**?**]).

$$\rho \left.\frac{\partial \boldsymbol{v}}{\partial t}\right|_{\boldsymbol{X}} = \nabla_x \cdot \boldsymbol{T} + \rho \boldsymbol{g} \quad , \quad \boldsymbol{T}^T = \boldsymbol{T} \tag{1.11}$$

In the same way we did before, transform the above equation of substance time derivative funciton $\partial/\partial t|X$ to the reference time derivative function $\partial/\partial t|\chi$ by using $f \equiv vi$ in Eq(2.11) の$f \equiv vi$. In respect, Euler representation of the ALE Cauchy's law of motion can be expressed by,

$$\rho \left.\frac{\partial \boldsymbol{v}}{\partial t}\right|_{\boldsymbol{\chi}} + \rho \boldsymbol{c} \cdot (\nabla_x \otimes \boldsymbol{v}) = \nabla_x \cdot \boldsymbol{T} + \rho \boldsymbol{g} \tag{2.19}$$

Written in its components, we have,

$$\rho \left.\frac{\partial v_i}{\partial t}\right|_{\boldsymbol{\chi}} + \rho c_j \frac{\partial v_i}{\partial x_j} = \frac{\partial \sigma_{ji}}{\partial x_j} + \rho g_i \quad (\text{in } R_x) \tag{2.20}$$

For the general transmission method of the flow force, there are two major laws that signify such case: Pascal's principle and Newton's Law of viscosity.

[1]*Pascal's Principle*

When pressure is put by single point of fluid in insulated container, the same amount of pressure is transmitted in the whole part.

[2]*Newton's Law of Viscosity*

A shearing stress occurred by viscosity is proportional to the velocity gradient, which perpendicular to the plane.

The formulation of the fact consists the commonly known Newton's constitutive equation for fluid. Provided that μ is a viscosity coefficient, and the kinetic viscosity coefficient is given $\nu \equiv \mu/\rho$.

$$\boldsymbol{T} = \left\{-p - \frac{2}{3}\mu(\mathrm{tr}\boldsymbol{D})\right\}\boldsymbol{I} + 2\mu\boldsymbol{D} \tag{2.21}$$

Furthermore, the flow we deal in this study, the compressibility is considered to be ignored, thus application of incompressibility condition $\mathrm{tr}\boldsymbol{D} = \nabla x \cdot \boldsymbol{v} = 0$ to the constitutive equation may simplify the equation.

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\mu\boldsymbol{D} \tag{2.22}$$

In component expression, we have,

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{2.23}$$

Generally , in fluid analysis, Newton's constitutive equation of fluid is applied to Cauchy's law of motion. Its transformation to the spatial time derivative function is called *Navier–Stokes* equation.

Now, the boundary condition of fluid should be consisted of the fundamental boundary condition(*Dirichlet boundary condition*) and natural boundary condition(*Neumann boundary condition*). They are established in the following.

$$v_i = \bar{v}_i \qquad (\text{on } \partial R_x^b) \tag{2.24}$$

$$\sigma_{ji}n_j = \bar{h}_i \qquad (\text{on } \partial R_x^h) \tag{2.25}$$

Yet, the boundary should satisfy the following.

$$\partial R_x = \overline{\partial R_x^b \cup \partial R_x^h} \;\;,\quad \varnothing = \partial R_x^b \cap \partial R_x^h \tag{2.26}$$

Deriving the finite element equation

Here, we aim to conduct finite element discritization of governing equation by creating a weak formulation. We further introduce generalization matrix and the vector to derive the matrix expression of its governing equation.

Deriving the weighted residual equation for governing equation

Until this point, we have derived the infinitesimal compressible ALE continuous system(2.18)and ALE Navier–Stokes equation(2.20). In the following, we conduct discritization of the governing equations of the two fluids by finite element method. We obtain the weak formulation by Galerkin method of the weighted residual method.(Huerta, A. and Liu, W. K. 1988[**?**]). Galerkin method (Bubnov–Glerkin method) is a technique of taking the weight function as shape function, and the formulation of further adding the functions other than shape functions to the weight functions is called Petrov–Glerkin method.

Here, the unknown state invariables are given to be *p* and the flow velocity ***v***, and define the weight functions that correspond to the respective variables as δp, and $\delta \boldsymbol{v}$. First, we conduct finite element discritization of the infinitesimal compressible ALE continuous system(2.18) by weighted residual method. *Rex* represents the element region in Euler representation.

$$\sum_e \int_{R_x^e} \delta p \frac{1}{B} \left.\frac{\partial p}{\partial t}\right|_{\boldsymbol{\chi}} \mathrm{d}R_x + \sum_e \int_{R_x^e} \delta p \frac{1}{B} c_i \frac{\partial p}{\partial x_i} \mathrm{d}R_x + \sum_e \int_{R_x^e} \delta p \frac{\partial v_i}{\partial x_i} \mathrm{d}R_x = 0 \tag{2.27}$$

Next, conduct finite element discritization of ALE Navier–Stokes equation (2.20) by weighted residual method to obtain the following.

$$\begin{aligned} &\sum_e \int_{R_x^e} \delta v_i \rho \left.\frac{\partial v_i}{\partial t}\right|_{\boldsymbol{\chi}} \mathrm{d}R_x + \sum_e \int_{R_x^e} \delta v_i \rho c_j \frac{\partial v_i}{\partial x_j} \mathrm{d}R_x \\ &\quad - \sum_e \int_{R_x^e} \delta v_i \frac{\partial \sigma_{ji}}{\partial x_j} \mathrm{d}R_x - \sum_e \int_{R_x^e} \delta v_i \rho g_i \, \mathrm{d}R_x = 0 \end{aligned} \tag{2.28}$$

Here considering Cauchy stress tensor of being symmetry, the following relational expression is introduced.

$$\begin{aligned} \nabla \cdot (\boldsymbol{\sigma} \cdot \delta \boldsymbol{v}) &= \frac{\partial(\sigma_{ij}\delta v_j)}{\partial x_i} = \delta v_j \frac{\partial \sigma_{ij}}{\partial x_i} + \frac{\partial \delta v_j}{\partial x_i}\sigma_{ij} \\ &= \delta v_i \frac{\partial \sigma_{ji}}{\partial x_j} + \frac{\partial \delta v_i}{\partial x_j}\sigma_{ji} \quad \left(= \delta \boldsymbol{v} \cdot (\nabla \cdot \boldsymbol{\sigma}) + \left((\nabla\, \delta \boldsymbol{v})^T\right) : \boldsymbol{\sigma}^T\right) \end{aligned} \tag{2.29}$$

If Gauss divergence theorem is applied in order to transform the volume integrals to the area integrals, we can write as,

$$\sum_e \int_{R_x^e} \nabla \cdot (\boldsymbol{\sigma} \cdot \delta \boldsymbol{v}) \, \mathrm{d}R_x = \int_{\partial R_x^h} \delta \boldsymbol{v} \cdot \boldsymbol{\sigma}^T \cdot \boldsymbol{n} \, \mathrm{d}S \tag{2.30}$$

Hence, Navier–Stokes discritization equation (2.28) of the third term related to the stress can be reformed as following.

$$-\sum_e \int_{R_x^e} \delta v_i \frac{\partial \sigma_{ji}}{\partial x_j} \mathrm{d}R_x = \sum_e \int_{R_x^e} \frac{\partial \delta v_i}{\partial x_j} \sigma_{ij} \, \mathrm{d}R_x - \int_{\partial R_x^h} \delta v_i \bar{h}_i \, \mathrm{d}S \tag{2.31}$$

The natural boundary condition equation (2.25)is being adopted in this case. When Newton's constitutional equation of fluid (2.23) is substituted,

$$\begin{aligned}
\frac{\partial \delta v_i}{\partial x_j}\sigma_{ji} &= \frac{\partial \delta v_i}{\partial x_j}\left\{-p\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right\} \\
&= -\frac{\partial \delta v_i}{\partial x_i}p + \mu\frac{\partial \delta v_i}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \\
&= -\frac{\partial \delta v_i}{\partial x_i}p + \frac{\mu}{2}\left(\frac{\partial \delta v_i}{\partial x_j} + \frac{\partial \delta v_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)
\end{aligned} \tag{2.32}$$

Thus, ALE Navier–Stokes discritization equation (2.28) can be eventually expressed by,

$$\begin{aligned}
&\sum_e \int_{R_x^e} \delta v_i \rho \left.\frac{\partial v_i}{\partial t}\right|_{\boldsymbol{\chi}} \mathrm{d}R_x + \sum_e \int_{R_x^e} \delta v_i \rho c_j \frac{\partial v_i}{\partial x_j}\,\mathrm{d}R_x \\
&\quad - \sum_e \int_{R_x^e} \frac{\partial \delta v_i}{\partial x_i} p\,\mathrm{d}R_x + \sum_e \int_{R_x^e} \frac{\mu}{2}\left(\frac{\partial \delta v_i}{\partial x_j} + \frac{\partial \delta v_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\mathrm{d}R_x \\
&\quad - \sum_e \int_{R_x^e} \delta v_i \rho g_i\,\mathrm{d}R_x - \int_{\partial R_x^h} \delta v_i \bar{h}_i\,\mathrm{d}S = 0
\end{aligned} \tag{2.33}$$

Production of matrix equations for fluid element

By conducting the weak formulation, finite elementary discritized infinitesimal compressible ALE continuous system(2.27)and ALE Navier–Stokes equation (2.33) are obtained. By introducing the given equations to generalization unknown variable vectors to express in matrix representation, the final formations should be the following.(Huerta, A. and Liu, W. K. 1988[**?**]).

微圧縮性 ALE 連続の式のマトリクス方程式

$$\boldsymbol{M}^P \cdot \overset{*}{\boldsymbol{p}} + \boldsymbol{\Lambda}^P \cdot \boldsymbol{p} + \boldsymbol{G}^T \cdot \boldsymbol{v} = \boldsymbol{0} \tag{2.34}$$

ALE Navier–Stokes の方程式のマトリクス方程式

$$\boldsymbol{M} \cdot \overset{*}{\boldsymbol{v}} + \boldsymbol{\Lambda} \cdot \boldsymbol{v} + \boldsymbol{K}_\mu \cdot \boldsymbol{v} - \boldsymbol{G} \cdot \boldsymbol{p} = \boldsymbol{F} \tag{2.35}$$

Provided that the symbol $*$ represents the time derivative (reference time derivative function $\partial / \partial t|$ffl)of unknown variables at ALE coordinates system. The two matrix equations can be put together to obtain single matrix equation by adopting the generalization vector ϕ at total unknown variables of the fluid.

流体解析のマトリクス方程式

$$\boldsymbol{M}_f \overset{*}{\boldsymbol{\varphi}} + \boldsymbol{C}_f \boldsymbol{\varphi} = \boldsymbol{F}_f \tag{2.36}$$

•Each matrix and the vector are written out in the following.

$$\boldsymbol{M}_f = \begin{bmatrix} \boldsymbol{M}^P & \\ & \boldsymbol{M} \end{bmatrix} \quad , \quad \boldsymbol{C}_f = \begin{bmatrix} \boldsymbol{\Lambda}^P & \boldsymbol{G}^T \\ -\boldsymbol{G} & \boldsymbol{\Lambda} + \boldsymbol{K}_\mu \end{bmatrix} \tag{2.37}$$

$$\boldsymbol{\varphi} = \begin{Bmatrix} \boldsymbol{p} \\ \boldsymbol{v} \end{Bmatrix} \quad , \quad \boldsymbol{F}_f = \begin{Bmatrix} 0 \\ \boldsymbol{F} \end{Bmatrix} \tag{2.38}$$

•Physical interpretations of the respective symbols are,

$$\left\{ \begin{array}{cl} \boldsymbol{M}^P, \boldsymbol{M} & : \text{圧力, 速度についての質量マトリクス} \\ \boldsymbol{\Lambda}^P, \boldsymbol{\Lambda} & : \text{圧力, 速度についての移流項 (対流項) のマトリクス} \\ \boldsymbol{F}^{\text{ext}} & : \text{外力ベクトル} \\ \boldsymbol{K}_\mu & : \text{粘性散逸マトリクス} \\ \boldsymbol{G} & : \text{発散についてのマトリクス} \\ \boldsymbol{p}^*, \boldsymbol{v}^* & : \boldsymbol{p}, \ \boldsymbol{v} \text{ の参照時間導関数} \end{array} \right. \tag{2.39}$$

•We define a symbol that represents the shape function of the fluid element as,

$$\left\{ \begin{array}{ll} N_a & : \text{節点 } a \text{ に関する速度の形状関数} \\ N_a^P & : \text{節点 } a \text{ に関する圧力の形状関数} \end{array} \right. \tag{2.40}$$

•The degree of freedom invariable used in finite element analysis is defined as,

$$\left\{ \begin{array}{ll} \text{1 要素内の圧力点数} & : \text{NEPN} \\ \text{1 要素内の速度点数} & : \text{NEN} \\ \text{空間の次元数} & : \text{NSD} \end{array} \right. \tag{2.41}$$

•Infinitesimal compressible **ALE** continuous system matrix

$$\boldsymbol{M}^P = \sum_e \boldsymbol{M}^{Pe} \quad , \quad \boldsymbol{M}^{Pe} = [M_{IJ}^P] \tag{2.42}$$

$$M_{IJ}^P = \int_{R_x^e} \frac{1}{B} N_a^P N_b^P \, \mathrm{d}R_x \tag{2.43}$$

$$\boldsymbol{\Lambda}^P = \sum_e \boldsymbol{\Lambda}^{Pe} \quad , \quad \boldsymbol{\Lambda}^{Pe} = [\Lambda_{IJ}^P] \tag{2.44}$$

$$\Lambda_{IJ}^P = \int_{R_x^e} \frac{1}{B} N_a^P c_k \frac{\partial N_b^P}{\partial x_k} \, \mathrm{d}R_x \tag{2.45}$$

$$\boldsymbol{G} = \sum_e \boldsymbol{G}^e \quad , \quad \boldsymbol{G}^e = [G_{MJ}] \tag{2.46}$$

$$G_{MJ} = \int_{R_x^e} N_b^P \frac{\partial N_c}{\partial x_m} \, \mathrm{d}R_x \tag{2.47}$$

•Where component index satisfies the following

$$\begin{cases} I = a \\ J = b \\ M = (c-1) \cdot \mathrm{NSD} + m \end{cases} \qquad \begin{cases} 1 \le a, b \le \mathrm{NEPN} \\ 1 \le c \le \mathrm{NEN} \\ 1 \le k, m \le \mathrm{NSD} \end{cases} \tag{2.48}$$

ALE Navier–Stokes equation

$$\boldsymbol{M} = \sum_e \boldsymbol{M}^e \quad , \quad \boldsymbol{M}^e = [M_{IJ}] \tag{2.49}$$

$$M_{IJ} = \int_{R_x^e} \delta_{ij} \rho N_a N_b \, \mathrm{d}R_x \tag{2.50}$$

$$\boldsymbol{\Lambda} = \sum_e \boldsymbol{\Lambda}^e \quad , \quad \boldsymbol{\Lambda}^e = [\Lambda_{IJ}] \tag{2.51}$$

$$\Lambda_{IJ} = \int_{R_x^e} \delta_{ij} \rho N_a c_m \frac{\partial N_b}{\partial x_m} \, \mathrm{d}R_x \tag{2.52}$$

$$\boldsymbol{F}^{\text{ext}} = \sum_e \boldsymbol{F}^{\text{ext}\,e} \quad , \quad \boldsymbol{F}^{\text{ext}\,e} = [F_I^{\text{ext}}] \tag{2.53}$$

$$F_I^{\text{ext}} = \int_{R_x^e} \rho N_a g_i \, \mathrm{d}R_x \tag{2.54}$$

$$\boldsymbol{K}^e = \int_{R_x^e} \boldsymbol{B}^T \cdot \boldsymbol{D} \cdot \boldsymbol{B} \, \mathrm{d}R_x \tag{2.55}$$

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_1 & \cdots & \boldsymbol{B}_a & \cdots & \boldsymbol{B}_{\text{NEN}} \end{bmatrix} \tag{2.56}$$

•Where component index satisfies the following

$$\begin{cases} I = (a-1) \cdot \text{NSD} + i \\ J = (b-1) \cdot \text{NSD} + j \end{cases} \qquad \begin{cases} 1 \le i, j, m \le \text{NSD} \\ 1 \le a, b \le \text{NEN} \end{cases} \tag{2.57}$$

•Under 3-D, matrix $\boldsymbol{B}$ and the matrix $\boldsymbol{D}$ may be given as:

$$\boldsymbol{B}_a^T = \begin{bmatrix} \frac{\partial N_a}{\partial x_1} & \frac{\partial N_a}{\partial x_2} & 0 & 0 & 0 & \frac{\partial N_a}{\partial x_3} \\ 0 & \frac{\partial N_a}{\partial x_1} & \frac{\partial N_a}{\partial x_2} & 0 & \frac{\partial N_a}{\partial x_3} & 0 \\ 0 & 0 & 0 & \frac{\partial N_a}{\partial x_3} & \frac{\partial N_a}{\partial x_2} & \frac{\partial N_a}{\partial x_1} \end{bmatrix} \tag{2.58}$$

$$\boldsymbol{D} = \begin{bmatrix} 2\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix} \tag{2.59}$$

In two dimension：

$$\boldsymbol{B}_a = \begin{bmatrix} \frac{\partial N_a}{\partial x_1} & 0 \\ \frac{\partial N_a}{\partial x_2} & \frac{\partial N_a}{\partial x_1} \\ 0 & \frac{\partial N_a}{\partial x_2} \end{bmatrix} \tag{2.60}$$

$$\boldsymbol{D} = \begin{bmatrix} 2\mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 2\mu \end{bmatrix} \tag{2.61}$$