Nonlinear
Finite Element Method

18/10/2004

Nonlinear Finite Element Method

- Lectures include discussion of the nonlinear finite element method.
- It is preferable to have completed “Introduction to Nonlinear Finite Element Analysis” available in summer session.
- If not, students are required to study on their own before participating this course. Reference:Toshiaki.,Kubo. “Introduction: Tensor Analysis For Nonlinear Finite Element Method” (Hisennkei Yugen Yoso no tameno Tensor Kaiseki no Kiso),Maruzen.
- Lecture references are available and downloadable at http://www.sml.k.u-tokyo.ac.jp/members/nabe/lecture2004 They should be put up on the website by the day before scheduled meeting day, and each students are expected to come in with a copy of the reference.
- Lecture notes from previous year are available and downloadable, also at http://www.sml.k.u.tokyo.ac.jp/members/nabe/lecture2003 You may find the course title, ”Advanced Finite Element Method” but the contents covered are almost the same I will cover this year.
- I will assign the exercises from this year, and expect the students to hand them in during the following lecture. They are not the requirements and they will not be graded, however it is important to actually practice calculate in deeper understanding the finite element method.
- For any questions, contact me at nabe@sml.k.u-tokyo.ac.jp

Nonlinear Finite Element Method Lecture Schedule

1. 10/ 4 Finite element analysis in boundary value problems and the differential equations
2. 10/18 Finite element analysis in linear elastic body
3. 10/25 Isoparametric solid element (program)
4. 11/ 1 Numerical solution and boundary condition processing for system of linear equations（with exercises）
5. 11/ 8 Basic program structure of the linear finite element method(program)
6. 11/15 Finite element formulation in geometric nonlinear problems(program)
7. 11/22 Static analysis technique、hyperelastic body and elastic-plastic material for nonlinear equations (program)
8. 11/29 Exercises for Lecture7
9. 12/ 6 Dynamic analysis technique and eigenvalue analysis in the nonlinear equations
10. 12/13 Structural element
11. 12/20 Numerical solution— skyline method、iterative method for the system of linear equations
12. 1/17 ALE finite element fluid analysis
13. 1/24 ALE finite element fluid analysis

Boundary Value Problems in Linear Elastic Body

Consider, a boundary value problem$[B]$ for a linear elastic body A found in the figure below. Ω is a region occupied by $[B]$, and the body A Ω has its boundary $\partial \Omega$. A displacement boundary condition is given on its subset $\partial \Omega D$. When surface force $\boldsymbol{t}$, body force $\rho\boldsymbol{g}$ are acted on such systems, find the displacement $\boldsymbol{u} \in V$ that satisfies the equilibrium condition. Density ρ, gravitational acceleration $\boldsymbol{g}$ and displacement V are considered as a set of all solution candidates that satisfy the admissible function for the displacements, or the displacement boundary condition, in other words.

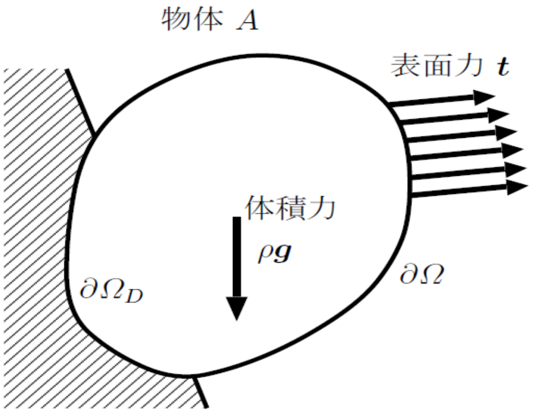

図 1: 境界値問題概念図

Linear elastic body obeys the Hooke's law. The microscopic transformation of such substance, the iron and the rubber, for example, are commonly known as isotropic, and its internal stress all depend on the displacement. The substance can be made a model.

Displacement boundary condition or the surface force are given at all points on the surface of substance $\partial \Omega$. Which implies the surface force is being provided at all points but $\partial \Omega D$. It is often omitted in a case in which the boundary value takes 0, therefore should be carefully observed.

Definition of Symbols

- We define a configuration of the substance at nominal time $t0$ as a nominal configuration, and express the position vector at each substance point as $\boldsymbol{X}$
- Position vector of a mass point $\boldsymbol{X}$ at the present time t is expressed as $\boldsymbol{x}$
- Displacement vector for the substance point from $t0$ to t is expressed as $\boldsymbol{u}$

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X} \tag{1}$$

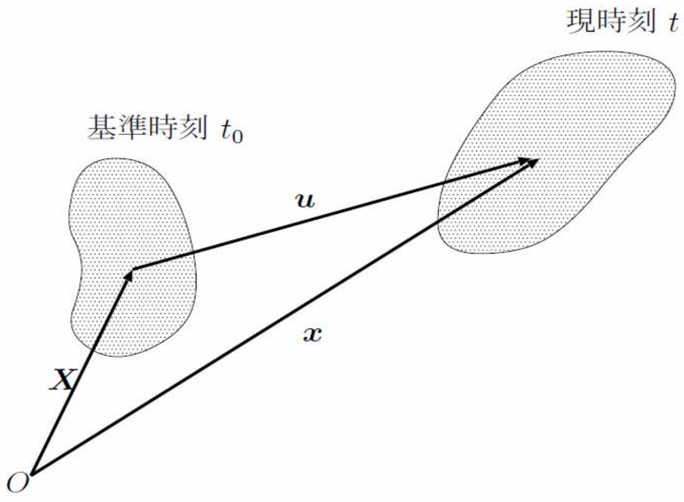

図 2: 物質点の運動

Strong Form1

This problems can be formulated by the following.
[*B*] Where ***t***, ***g*** are given, find ***u*** ∈ *V* that satisfies the following:
[1] Balance equation(Cauchy’ equation of motion)

$$\nabla_x \cdot \boldsymbol{T} + \rho \boldsymbol{g} = 0 \tag{2}$$

[2] Boundary condition equation

$$\boldsymbol{u} = \underline{\boldsymbol{u}} \quad \text{on} \quad \partial\Omega_D \tag{3}$$

$$\boldsymbol{T}^T \cdot \boldsymbol{n} = \boldsymbol{t} \quad \text{on} \quad \partial\Omega - \partial\Omega_D \tag{4}$$

[3] Displacement･strain relational expression

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \tag{5}$$

[4] Stress･strain relational expression(constructive equation)

$$T_{ij} = \kappa(\mathrm{div}\,\boldsymbol{u})\delta_{ij} + 2G\varepsilon_{ij}^{D}(\boldsymbol{u}) \tag{6}$$

- In any problems, [1] and [2] are congruent. (possibly reformed in equivalent expressions if necessary.)

[4] depends on its substance model, and [3] is determined in correspond to [4]

Definition of Symbols

This problem can be formulated as in the following:

[*B*] With given ***t*** and ***g***, obtain ***u*** $\in$ *V* that satisfies the following equations.

$$\nabla_x \cdot \boldsymbol{T} + \rho \boldsymbol{g} = 0 \tag{7}$$

$$\boldsymbol{T}^T \cdot \boldsymbol{n} = \boldsymbol{t} \tag{8}$$

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \tag{9}$$

$$T_{ij} = \kappa(\mathrm{div}\, \boldsymbol{u})\delta_{ij} + 2G\varepsilon^D_{ij}(\boldsymbol{u}) \tag{10}$$

- A set of all admissible function of the displacement *V*
- ***T*** Cauchy stress
- κ, *G* bulk modulus, modulus of rigidity (physical property)
- δ*ij* Kronecker delta symbol

$$\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{11}$$

- εij , εDij linear strain, deviator strain

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \tag{12}$$

$$\varepsilon_{ij}^{D}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) - \frac{1}{3}(\operatorname{div}\boldsymbol{u})\delta_{ij} \tag{13}$$

$$\operatorname{div}\boldsymbol{u} = \frac{\partial u_i}{\partial X_i} = \frac{\partial u_1}{\partial X_1} + \frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_3} = \operatorname{tr}\left(\varepsilon_{ij}\right) \tag{14}$$

Weak Form

- As we stated earlier, the finite element method is associated with the approximate analysis of the weak form of the differential equations.
- [*V*] represents the weak form corresponding to [*B*].

[*V*] With the surface force $\boldsymbol{t}$ and the body force $\rho\boldsymbol{g}$ given, obtain $\boldsymbol{u} \in V$ that satisfies the following.

$$\begin{aligned}
&\int_{\Omega} T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u})\,\mathrm{d}\Omega \\
&= \int_{\partial\Omega} \delta\boldsymbol{u}\cdot\boldsymbol{t}\,\mathrm{d}S + \int_{\Omega} \rho\,\delta\boldsymbol{u}\cdot\boldsymbol{g}\,\mathrm{d}\Omega \qquad \forall\delta\boldsymbol{u}\in\mathcal{V} \qquad (15)
\end{aligned}$$

- summation convention is used for $Tij(\boldsymbol{v})\ \varepsilon ij(\delta\boldsymbol{u})$

- Therefore,

$$\begin{aligned} T_{ij}(\boldsymbol{v})\,\varepsilon_{ij}(\boldsymbol{v}) &= \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}(\boldsymbol{v})\,\varepsilon_{ij}(\boldsymbol{v}) \\ = \quad & T_{11}(\boldsymbol{v})\,\varepsilon_{11}(\boldsymbol{v}) + T_{12}(\boldsymbol{v})\,\varepsilon_{12}(\boldsymbol{v}) + T_{13}(\boldsymbol{v})\,\varepsilon_{13}(\boldsymbol{v}) \\ & + T_{21}(\boldsymbol{v})\,\varepsilon_{21}(\boldsymbol{v}) + T_{22}(\boldsymbol{v})\,\varepsilon_{22}(\boldsymbol{v}) + T_{23}(\boldsymbol{v})\,\varepsilon_{23}(\boldsymbol{v}) \\ & + T_{31}(\boldsymbol{v})\,\varepsilon_{31}(\boldsymbol{v}) + T_{32}(\boldsymbol{v})\,\varepsilon_{32}(\boldsymbol{v}) + T_{33}(\boldsymbol{v})\,\varepsilon_{33}(\boldsymbol{v}) \qquad (16) \end{aligned}$$

Strong Form, Weak Form, Stationary Potential Energy Principles

- In the previous lecture, we conducted regional integrations by multiplying $\boldsymbol{v} \in V$ to the strong form [*B*] by both sides to derive the weak form [*V*], and they became discretized by introducing the finite element.
- We also introduced the minimization problem of the potential energy [*M*] to the equivalent formulation [*V*]
- In dealing with boundary value problems of the linear elastic body, formulation is often used based on [*M*].
- The reasons for above may be considered as:

– There are cases where the material models are defined by the potentials.

– Under the presence of the potentials, the stiffness matrix may be found in symmetry. (while complex elastic body and fluid are found in the absence of the potentials)

– If there is a subsidiary condition, Lagrange multiplier and the penalty method may be introduced to handle.

Stationary Potential Energy Principle

[*M*] Given ***t*** and ***g***, obtain ***u*** $\in V$ that satisfies the following

$$\mathbf{\Pi}(\boldsymbol{u}) \leq \mathbf{\Pi}(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \mathcal{V} \tag{17}$$

Provided that,

$$\begin{aligned}\mathbf{\Pi}(\boldsymbol{v}) &= \int_{\Omega} \frac{1}{2} T_{ij}(\boldsymbol{v})\, \varepsilon_{ij}(\boldsymbol{v})\, \mathrm{d}\Omega - \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{t}\, \mathrm{d}S - \int_{\Omega} \rho \boldsymbol{v} \cdot \boldsymbol{g}\, \mathrm{d}\Omega \\ &= \frac{\kappa}{2} \int_{\Omega} (\mathrm{div}\, \boldsymbol{v})^2\, \mathrm{d}\Omega + G \int_{\Omega} \varepsilon_{ij}^{D}(\boldsymbol{v})\, \varepsilon_{ij}^{D}(\boldsymbol{v})\, \mathrm{d}\Omega - \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{t}\, \mathrm{d}S - \int_{\Omega} \rho \boldsymbol{v} \cdot \boldsymbol{g}\, \mathrm{d}\Omega\end{aligned} \tag{18}$$

[*V*] Given the surface force ***t* and the** body force ρ***g***, obtain ***u*** $\in V$,which satisfies the following.

$$\begin{aligned}\int_{\Omega} T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u})\,\mathrm{d}\Omega &\\ &= \kappa\int_{\Omega}(\operatorname{div}\boldsymbol{u})(\operatorname{div}\delta\boldsymbol{u})\,\mathrm{d}\Omega + 2G\int_{\Omega}\varepsilon_{ij}^{D}(\boldsymbol{u})\varepsilon_{ij}^{D}(\delta\boldsymbol{u})\,\mathrm{d}\Omega \\ &= \int_{\partial\Omega}\delta\boldsymbol{u}\cdot\boldsymbol{t}\,\mathrm{d}S + \int_{\Omega}\rho\,\delta\boldsymbol{u}\cdot\boldsymbol{g}\,\mathrm{d}\Omega \qquad \forall\delta\boldsymbol{u}\in\mathcal{V} \quad (19)\end{aligned}$$

$[M] \Rightarrow [V]$ 1

- The equation $[M]$ establishes the following

$$\mathbf{\Pi}(\boldsymbol{v}) - \mathbf{\Pi}(\boldsymbol{u}) \geq 0 \qquad \forall \boldsymbol{v} \in \mathcal{V} \tag{20}$$

- Then $\delta \boldsymbol{u} \in V\ (\boldsymbol{v} = \delta \boldsymbol{u} + \boldsymbol{u})$ exists for the arbitrary $\boldsymbol{v}$
- Whereas,

$$\mathbf{\Pi}(\delta\boldsymbol{u} + \boldsymbol{u}) - \mathbf{\Pi}(\boldsymbol{u}) \geq 0 \qquad \forall \delta\boldsymbol{u} \in \mathcal{V} \tag{21}$$

- We may yield the following equations from above,

$$\begin{aligned}&\left\{\frac{\kappa}{2}\int_\Omega \mathrm{div}\,(\delta\boldsymbol{u}+\boldsymbol{u})\,\mathrm{div}\,(\delta\boldsymbol{u}+\boldsymbol{u})\,\mathrm{d}\Omega + G\int_\Omega \varepsilon_{ij}^D(\delta\boldsymbol{u}+\boldsymbol{u})\,\varepsilon_{ij}^D(\delta\boldsymbol{u}+\boldsymbol{u})\,\mathrm{d}\Omega - \int_{\partial\Omega}\boldsymbol{t}\cdot(\delta\boldsymbol{u}+\boldsymbol{u})\,\mathrm{d}S - \int_\Omega \rho\boldsymbol{g}\cdot(\delta\boldsymbol{u}+\boldsymbol{u})\,\mathrm{d}\Omega\right\} \\ &- \left\{\frac{\kappa}{2}\int_\Omega (\mathrm{div}\,\boldsymbol{u})(\mathrm{div}\,\boldsymbol{u})\,\mathrm{d}\Omega - G\int_\Omega \varepsilon_{ij}^D(\boldsymbol{u})\,\varepsilon_{ij}^D(\boldsymbol{u})\,\mathrm{d}\Omega - \int_{\partial\Omega}\boldsymbol{t}\cdot\boldsymbol{u}\,\mathrm{d}S - \int_\Omega \rho\boldsymbol{g}\cdot\boldsymbol{u}\,\mathrm{d}\Omega\right\} \geq 0 \qquad \forall\delta\boldsymbol{u}\in\mathcal{V}\end{aligned} \tag{22}$$

- Only if the following equations are used.

$$\mathrm{div}(\delta\boldsymbol{u} + \boldsymbol{u}) = \mathrm{div}(\delta\boldsymbol{u}) + \mathrm{div}(\boldsymbol{u}) \tag{23}$$

$$\mathrm{div}(\boldsymbol{u}) = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \tag{24}$$

$$\begin{aligned}\mathrm{div}(\boldsymbol{u} + \delta\boldsymbol{u}) &= \frac{\partial(u_1 + \delta u_1)}{\partial x_1} + \frac{\partial(u_2 + \delta u_2)}{\partial x_2} + \frac{\partial(u_3 + \delta u_3)}{\partial x_3} \\ &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} + \frac{\partial \delta u_1}{\partial x_1} + \frac{\partial \delta u_2}{\partial x_2} + \frac{\partial \delta u_3}{\partial x_3} \\ &= \mathrm{div}(\boldsymbol{u}) + \mathrm{div}(\delta\boldsymbol{u})\end{aligned} \tag{25}$$

$[M] \Rightarrow [V]$ 2

- To put in order,

$$\begin{aligned}\kappa \int_{\Omega} (\operatorname{div} \boldsymbol{u})(\operatorname{div} \delta\boldsymbol{u})\, \mathrm{d}\Omega + 2G \int_{\Omega} \varepsilon_{ij}^{D}(\boldsymbol{u})\, \varepsilon_{ij}^{D}(\delta\boldsymbol{u})\, \mathrm{d}\Omega & \\ - \int_{\partial\Omega} \delta\boldsymbol{u} \cdot \boldsymbol{t}\, \mathrm{d}S - \int_{\Omega} \rho\, \delta\boldsymbol{u} \cdot \boldsymbol{g}\, \mathrm{d}\Omega & \\ + \frac{\kappa}{2} \int_{\Omega} (\operatorname{div} \delta\boldsymbol{u})(\operatorname{div} \delta\boldsymbol{u})\, \mathrm{d}\Omega & \\ + G \int_{\Omega} \varepsilon_{ij}^{D}(\delta\boldsymbol{u})\, \varepsilon_{ij}^{D}(\delta\boldsymbol{u})\, \mathrm{d}\Omega \geq 0 \qquad \forall \delta\boldsymbol{u} \in \mathcal{V} & \quad (26)\end{aligned}$$

- Since the equations above must be established with the arbitrary $\delta\boldsymbol{u}$, we must show, by substituting $\alpha\, \delta\boldsymbol{u}$ into $\delta\boldsymbol{u}$ **,** the equations are established with the arbitrary scalar α as well.

$$\begin{aligned}&\alpha \left\{ \kappa \int_{\Omega} (\operatorname{div} \boldsymbol{u})(\operatorname{div} \delta\boldsymbol{u})\, \mathrm{d}\Omega + 2G \int_{\Omega} \varepsilon_{ij}^{D}(\boldsymbol{u})\, \varepsilon_{ij}^{D}(\delta\boldsymbol{u})\, \mathrm{d}\Omega - \int_{\partial\Omega} \delta\boldsymbol{u} \cdot \boldsymbol{t}\, \mathrm{d}S - \int_{\Omega} \rho\, \delta\boldsymbol{u} \cdot \boldsymbol{g}\, \mathrm{d}\Omega \right\} \\ &+ \alpha^{2} \left\{ \frac{\kappa}{2} \int_{\Omega} (\operatorname{div} \delta\boldsymbol{u})(\operatorname{div} \delta\boldsymbol{u})\, \mathrm{d}\Omega + G \int_{\Omega} \varepsilon_{ij}^{D}(\delta\boldsymbol{u})\, \varepsilon_{ij}^{D}(\delta\boldsymbol{u})\, \mathrm{d}\Omega \right\} \geq 0 \qquad \forall \delta\boldsymbol{u} \in \mathcal{V} \qquad (27)\end{aligned}$$

$[M] \Rightarrow [V]$ 3

- The left hand side becomes 0 when $\delta\boldsymbol{u} \equiv 0$, thus it is verified.

$$\alpha \left\{ \kappa \int_\Omega (\operatorname{div} \boldsymbol{u})(\operatorname{div} \delta\boldsymbol{u}) \, \mathrm{d}\Omega + 2G \int_\Omega \varepsilon_{ij}^D(\boldsymbol{u}) \, \varepsilon_{ij}^D(\delta\boldsymbol{u}) \, \mathrm{d}\Omega - \int_{\partial\Omega} \delta\boldsymbol{u} \cdot \boldsymbol{t} \, \mathrm{d}S - \int_\Omega \rho \, \delta\boldsymbol{u} \cdot \boldsymbol{g} \, \mathrm{d}\Omega \right\}$$
$$+ \alpha^2 \left\{ \frac{\kappa}{2} \int_\Omega (\operatorname{div} \delta\boldsymbol{u})(\operatorname{div} \delta\boldsymbol{u}) \, \mathrm{d}\Omega + G \int_\Omega \varepsilon_{ij}^D(\delta\boldsymbol{u}) \, \varepsilon_{ij}^D(\delta\boldsymbol{u}) \, \mathrm{d}\Omega \right\} \geq 0 \qquad \forall \delta\boldsymbol{u} \in \mathcal{V} \tag{28}$$

- The equation above is considered as quadratic of α when $\delta\boldsymbol{u} \equiv 0$
- Therefore,

$$\begin{aligned}
y &= a\alpha^2 + b\alpha + c \\
a &= \left\{ \frac{\kappa}{2} \int_\Omega (\operatorname{div} \delta\boldsymbol{u})(\operatorname{div} \delta\boldsymbol{u}) \, \mathrm{d}\Omega + G \int_\Omega \varepsilon_{ij}^D(\delta\boldsymbol{u}) \, \varepsilon_{ij}^D(\delta\boldsymbol{u}) \, \mathrm{d}\Omega \right\} \\
b &= \left\{ \kappa \int_\Omega (\operatorname{div} \boldsymbol{u})(\operatorname{div} \delta\boldsymbol{u}) \, \mathrm{d}\Omega + 2G \int_\Omega \varepsilon_{ij}^D(\boldsymbol{u}) \, \varepsilon_{ij}^D(\delta\boldsymbol{u}) \, \mathrm{d}\Omega \right. \\
&\quad \left. - \int_{\partial\Omega} \delta\boldsymbol{u} \cdot \boldsymbol{t} \, \mathrm{d}S - \int_\Omega \rho \, \delta\boldsymbol{u} \cdot \boldsymbol{g} \, \mathrm{d}\Omega \right\} \\
c &= 0
\end{aligned} \tag{29}$$

$[M] \Rightarrow [V]$ 4

- A necessary and sufficient condition for $y \geq 0$ should be $a > 0$ and $b2 - 4ac \leq 0$ (discriminant), yet in this case the condition should be $b = 0$.
- Therefore, $[V]$ becomes the necessary and sufficient condition.

$$
\begin{aligned}
&\int_{\Omega} T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u})\,\mathrm{d}\Omega \\
&\qquad = \kappa \int_{\Omega} (\operatorname{div}\boldsymbol{u})(\operatorname{div}\delta\boldsymbol{u})\,\mathrm{d}\Omega + 2G \int_{\Omega} \varepsilon^{D}_{ij}(\boldsymbol{u})\varepsilon^{D}_{ij}(\delta\boldsymbol{u})\,\mathrm{d}\Omega \\
&\qquad = \int_{\partial\Omega} \delta\boldsymbol{u}\cdot\boldsymbol{t}\,\mathrm{d}S + \int_{\Omega} \rho\,\delta\boldsymbol{u}\cdot\boldsymbol{g}\,\mathrm{d}\Omega \qquad \forall\delta\boldsymbol{u} \in \mathcal{V} \quad (30)
\end{aligned}
$$

$[V] \Rightarrow [M]$

- To demonstrate $[V] \Rightarrow [M]$, notice the underlined parts to be 0.

$$
\begin{aligned}
&\Pi(\delta \boldsymbol{u} + \boldsymbol{u}) - \Pi(\boldsymbol{u}) \\
&= \underline{\kappa \int_{\Omega} (\operatorname{div} \boldsymbol{u})(\operatorname{div} \delta \boldsymbol{u}) \, \mathrm{d}\Omega + 2G \int_{\Omega} \varepsilon_{ij}^{D}(\boldsymbol{u}) \, \varepsilon_{ij}^{D}(\delta \boldsymbol{u}) \, \mathrm{d}\Omega} \\
&\underline{- \int_{\partial\Omega} \delta \boldsymbol{u} \cdot \boldsymbol{t} \, \mathrm{d}S - \int_{\Omega} \rho \, \delta \boldsymbol{u} \cdot \boldsymbol{g} \, \mathrm{d}\Omega} \\
&+ \frac{\kappa}{2} \int_{\Omega} (\operatorname{div} \delta \boldsymbol{u})(\operatorname{div} \delta \boldsymbol{u}) \, \mathrm{d}\Omega + G \int_{\Omega} \varepsilon_{ij}^{D}(\delta \boldsymbol{u}) \, \varepsilon_{ij}^{D}(\delta \boldsymbol{u}) \, \mathrm{d}\Omega \\
&= \frac{\kappa}{2} \int_{\Omega} (\operatorname{div} \delta \boldsymbol{u})(\operatorname{div} \delta \boldsymbol{u}) \, \mathrm{d}\Omega + G \int_{\Omega} \varepsilon_{ij}^{D}(\delta \boldsymbol{u}) \, \varepsilon_{ij}^{D}(\delta \boldsymbol{u}) \, \mathrm{d}\Omega \\
&\geq 0 \qquad \forall \delta \boldsymbol{u} \in \mathcal{V}
\end{aligned}
\tag{31}
$$

$[V] \Rightarrow [B]$ 1

- Using the symmetric property of $Tij(\boldsymbol{u})$ about i, j,

$$\begin{aligned}\int_{\Omega} T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u})\,\mathrm{d}\Omega &= \int_{\Omega} T_{ij}\cdot\frac{1}{2}\left(\frac{\partial\delta u_i}{\partial x_j}+\frac{\partial\delta u_j}{\partial x_i}\right)\mathrm{d}\Omega\\ &= \int_{\Omega} T_{ij}\frac{\partial\delta u_i}{\partial x_j}\,\mathrm{d}\Omega\end{aligned} \tag{32}$$

In this microscopic transformation, we take advantage of ∂ui to obtain . Apparently, this substitution is possible only with the microscopic transformations, and cannot be applied to the finite transformations.

- Integration by parts.

$$\int_{\Omega} T_{ij}\frac{\partial\delta u_i}{\partial x_j}\,\mathrm{d}\Omega = \int_{\Omega}\frac{\partial}{\partial x_j}(T_{ij}\delta u_i)\,\mathrm{d}\Omega - \int_{\Omega}\frac{\partial T_{ij}}{\partial x_j}\delta u_i\,\mathrm{d}\Omega \tag{33}$$

- Apply the Gauss' theorem to the left hand side of the first term in the equation above, then we obtain the following.(for the right hand side in the second term, we use $Tij = Tji$)

$$\int_{\Omega} T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u})\,\mathrm{d}\Omega = \int_{\partial\Omega} n_j(T_{ij}\delta u_i)\,\mathrm{d}S - \int_{\Omega}(\nabla_x\cdot\boldsymbol{T})_i\delta u_i\,\mathrm{d}\Omega \tag{34}$$

- Gauss' theorem (divergence theorem)(given $\boldsymbol{n}$ represents the normal vector on the boundary points)

$$\begin{aligned}\int_{\Omega}\mathrm{div}\,\boldsymbol{b}d\Omega &= \int_{\partial\Omega}\boldsymbol{n}\cdot\boldsymbol{b}dS\\ \int_{\Omega}\frac{\partial b_i}{\partial X_i}d\Omega &= \int_{\partial\Omega} n_i\, b_i dS\end{aligned} \tag{35}$$

$[V] \Rightarrow [B]$ 2

- Together with the external force term, we may write,

$$\int_{\Omega} T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u})\,\mathrm{d}\Omega - \int_{\partial\Omega} \delta\boldsymbol{u}\cdot\boldsymbol{t}\,\mathrm{d}S - \int_{\Omega} \rho\,\delta\boldsymbol{u}\cdot\boldsymbol{g}\,\mathrm{d}\Omega = 0 \tag{36}$$

$$\Leftrightarrow \int_{\partial\Omega} n_j(T_{ij}\delta u_i)\,\mathrm{d}S - \int_{\Omega} (\nabla_x\cdot\boldsymbol{T})_i\delta u_i\,\mathrm{d}\Omega - \int_{\partial\Omega} \delta\boldsymbol{u}\cdot\boldsymbol{t}\,\mathrm{d}S - \int_{\Omega} \rho\,\delta\boldsymbol{u}\cdot\boldsymbol{g}\,\mathrm{d}\Omega = 0 \tag{37}$$

$$\Leftrightarrow \int_{\Omega} \delta u_i(\nabla_x\cdot\boldsymbol{T} + \rho\boldsymbol{g})_i\,\mathrm{d}\Omega + \int_{\partial\Omega} \delta u_i(\boldsymbol{T}^T\cdot\boldsymbol{n} - \boldsymbol{t})_i\,\mathrm{d}S = 0 \tag{38}$$

- The condition for the equations above should be,

$$\nabla_x\cdot\boldsymbol{T} + \rho\boldsymbol{g} = 0 \tag{39}$$

$$\boldsymbol{T}^T\cdot\boldsymbol{n} = \boldsymbol{t} \tag{40}$$

(we use $T_{ij} = T_{ji}$)

- $[B] \Rightarrow [V]$ may be verified with taking the reverse steps.

Finite Element Formulation

- So far, we have examined the boundary value problems in the linear elastic body $[V]$ to be formulated in the weak form. Consider now for the finite element formulation based on the fact.

$[V]$ Given the external surface force $\boldsymbol{t}$ and the body force $\rho\boldsymbol{g}$, obtain $\boldsymbol{u} \in V$ that satisfies the following condition.

$$\int_{\Omega} T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u})\,\mathrm{d}\Omega = \int_{\partial\Omega} \delta\boldsymbol{u}\cdot\boldsymbol{t}\,\mathrm{d}S + \int_{\Omega} \rho\,\delta\boldsymbol{u}\cdot\boldsymbol{g}\,\mathrm{d}\Omega \qquad \forall\delta\boldsymbol{u}\in\mathcal{V} \tag{41}$$

Finite Element Subdivision and Interpolation 1

- In the finite element method, the region Ω , the analysis object is divided in the elements with the finite magnitude. Which is expressed in the following formulation,

$$\Omega = \sum_e \Omega_e \tag{42}$$

- Therefore, the regional integration along with the boundary integration may be gained by:

$$\int_\Omega \mathrm{d}\Omega = \sum_e \int_{\Omega_e} \mathrm{d}\Omega \tag{43}$$

$$\int_{\partial\Omega} \mathrm{d}S = \sum_e \int_{\partial\Omega_e} \mathrm{d}S \tag{44}$$

- Thus, the weak form is being modified (from now on we denote as $[Ve]$)

$$\sum_e \Bigg[\int_{\Omega_e} T_{ij}(\boldsymbol{u})\, \varepsilon_{ij}(\delta\boldsymbol{u})\, \mathrm{d}\Omega - \int_{\partial\Omega_e} \delta\boldsymbol{u}\cdot\boldsymbol{t}\, \mathrm{d}S - \int_{\Omega_e} \rho\, \delta\boldsymbol{u}\cdot\boldsymbol{g}\, \mathrm{d}\Omega \Bigg] = 0 \qquad \forall \delta\boldsymbol{u} \in \mathcal{V} \tag{45}$$

Finite Element Subdivision and Interpolation 2

- We assume $\boldsymbol{x}$ and $\boldsymbol{u}$ included in the integrand to be expressed by the interpolation functions within each element.

$$x_i = N^{(j)} x_i^{(j)} \tag{46}$$

$$u_i = N^{(j)} u_i^{(j)} \tag{47}$$

- We will cover the interpolation functions in the following lecture.

Matrix Notation

- We utilize the matrix notations for the convenience in the calculations.
- The matrix notations we show in the following are fundamentally introduced as a procedural means, and which contains no intrinsic implications, therefore, each programmer may arrange his/her own way to meet the needs.
- We introduce the most common and applicable procedures in the following.

Stress-Strain Matrix ($[D]$ Matrix) 1

- $[Ve]$ The integrands $T_{ij}(\boldsymbol{u})\ \varepsilon_{ij}(\delta\boldsymbol{u})$ in the left hand side in the first term may be expressed as the following if the summation convention was not being used.

$$\begin{aligned}T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u}) = \quad & T_{11}(\boldsymbol{u})\,\varepsilon_{11}(\delta\boldsymbol{u}) + T_{12}(\boldsymbol{u})\,\varepsilon_{12}(\delta\boldsymbol{u}) + T_{13}(\boldsymbol{u})\,\varepsilon_{13}(\delta\boldsymbol{u}) \\ & + T_{21}(\boldsymbol{u})\,\varepsilon_{21}(\delta\boldsymbol{u}) + T_{22}(\boldsymbol{u})\,\varepsilon_{22}(\delta\boldsymbol{u}) + T_{23}(\boldsymbol{u})\,\varepsilon_{23}(\delta\boldsymbol{u}) \\ & + T_{31}(\boldsymbol{u})\,\varepsilon_{31}(\delta\boldsymbol{u}) + T_{32}(\boldsymbol{u})\,\varepsilon_{32}(\delta\boldsymbol{u}) + T_{33}(\boldsymbol{u})\,\varepsilon_{33}(\delta\boldsymbol{u}) \qquad (48)\end{aligned}$$

- Using the symmetry property of T_{ij} and ε_{ij} about I and j, organize the equations in order to have the least operation times.

$$\begin{aligned}& T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u}) \\ & = \quad T_{11}(\boldsymbol{u})\,\varepsilon_{11}(\delta\boldsymbol{u}) + \ T_{22}(\boldsymbol{u})\,\varepsilon_{22}(\delta\boldsymbol{u}) + \ T_{33}(\boldsymbol{u})\,\varepsilon_{33}(\delta\boldsymbol{u}) \\ & + 2T_{12}(\boldsymbol{u})\,\varepsilon_{12}(\delta\boldsymbol{u}) + 2T_{23}(\boldsymbol{u})\,\varepsilon_{23}(\delta\boldsymbol{u}) + 2T_{31}(\boldsymbol{u})\,\varepsilon_{31}(\delta\boldsymbol{u}) \\ & = \{\varepsilon(\delta\boldsymbol{u})\}^T\{T(\boldsymbol{u})\} \qquad (49)\end{aligned}$$

- $\{\varepsilon(\boldsymbol{v})\}$, $\{T(\boldsymbol{v})\}$ is defined by the following equations.

$$\{\varepsilon(\boldsymbol{v})\} = \left\{\begin{matrix} \varepsilon_{11}(\boldsymbol{v}) \\ \varepsilon_{22}(\boldsymbol{v}) \\ \varepsilon_{33}(\boldsymbol{v}) \\ 2\varepsilon_{12}(\boldsymbol{v}) \\ 2\varepsilon_{23}(\boldsymbol{v}) \\ 2\varepsilon_{31}(\boldsymbol{v}) \end{matrix}\right\}, \quad \{T(\boldsymbol{v})\} = \left\{\begin{matrix} T_{11}(\boldsymbol{v}) \\ T_{22}(\boldsymbol{v}) \\ T_{33}(\boldsymbol{v}) \\ T_{12}(\boldsymbol{v}) \\ T_{23}(\boldsymbol{v}) \\ T_{31}(\boldsymbol{v}) \end{matrix}\right\} \qquad (50)$$

Stress-**Strain Matrix(**$[D]$ Matrix**)** 2

- Relational expression for the stress Tij and the strain εij can be,

$$T_{ij} = \kappa(\operatorname{div} \boldsymbol{u})\delta_{ij} + 2G\varepsilon_{ij}^{D}(\boldsymbol{u}) \tag{51}$$

Based on the relational expression, have $\{T(\boldsymbol{v})\}$ and $\{\varepsilon(\boldsymbol{v})\}$ correlate with the matrix and the vector product formulations.

$$\{T(\boldsymbol{v})\} = [D]\{\varepsilon(\boldsymbol{v})\} \tag{52}$$

- This matrix $[D]$ is often called the stress-strain matrix, or simply called $[D]$ matrix.
- We can write out the components of Tij found in $\{Tij(\boldsymbol{v})\}$.

$$T_{11} = \kappa\,(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2G\varepsilon_{11} - \frac{2G}{3}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \tag{53}$$

$$T_{22} = \kappa\,(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2G\varepsilon_{22} - \frac{2G}{3}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \tag{54}$$

$$T_{33} = \kappa\,(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2G\varepsilon_{33} - \frac{2G}{3}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \tag{55}$$

$$T_{12} = 2G\varepsilon_{12} \tag{56}$$

$$T_{23} = 2G\varepsilon_{23} \tag{57}$$

$$T_{31} = 2G\varepsilon_{31} \tag{58}$$

Stress-Strain Matrix([*D*] Matrix) 3

- It might look a little pressing to bring then into the matrix expressions though, we obtain the following.

$$\begin{Bmatrix} T_{11}\\ T_{22}\\ T_{33}\\ T_{12}\\ T_{23}\\ T_{31} \end{Bmatrix} = \begin{bmatrix} \kappa & \kappa & \kappa & & & \\ \kappa & \kappa & \kappa & & 0 & \\ \kappa & \kappa & \kappa & & & \\ & & & & & \\ & 0 & & & & \\ & & & & & \end{bmatrix} \begin{Bmatrix} \varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{23}\\ 2\varepsilon_{31} \end{Bmatrix} + \begin{bmatrix} \frac{4}{3}G & -\frac{2}{3}G & -\frac{2}{3}G & & & \\ -\frac{2}{3}G & \frac{4}{3}G & -\frac{2}{3}G & & 0 & \\ -\frac{2}{3}G & -\frac{2}{3}G & \frac{4}{3}G & & & \\ & & & G & & \\ & 0 & & & G & \\ & & & & & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ 2\varepsilon_{12}\\ 2\varepsilon_{23}\\ 2\varepsilon_{31} \end{Bmatrix} \tag{59}$$

- Now we define [*Dv*], [*Dd*] in the next step.

$$[D_v] = \begin{bmatrix} \kappa & \kappa & \kappa & & & \\ \kappa & \kappa & \kappa & & 0 & \\ \kappa & \kappa & \kappa & & & \\ & & & & & \\ & 0 & & & & \\ & & & & & \end{bmatrix} \quad [D_d] = \begin{bmatrix} \frac{4}{3}G & -\frac{2}{3}G & -\frac{2}{3}G & & & \\ -\frac{2}{3}G & \frac{4}{3}G & -\frac{2}{3}G & & 0 & \\ -\frac{2}{3}G & -\frac{2}{3}G & \frac{4}{3}G & & & \\ & & & G & & \\ & 0 & & & G & \\ & & & & & G \end{bmatrix} \tag{60}$$

Stress-Strain Matrix ($[D]$ Matrix) 4

- Using the matrix notation obtained in above, $[D]$ is defined by

$$[D] = [D_v] + [D_d] \tag{61}$$

- Furthermore, the integrands $Tij(\boldsymbol{u})$ $\varepsilon ij(\delta \boldsymbol{u})$ found on the left hand side in the first term $[Ve]$ can be expressed as,

$$\begin{aligned} T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u}) &= \{\varepsilon(\delta\boldsymbol{u})\}^T\,\{\boldsymbol{T}(\boldsymbol{u})\} \\ &= \{\varepsilon(\delta\boldsymbol{u})\}^T\,[D]\,\{\varepsilon(\boldsymbol{u})\} \end{aligned} \tag{62}$$

Node Displacement- Strain Matrix (*[B]* Matrix) 1

- Displacement and linear strain

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \tag{63}$$

- Displacement and the node displacement

$$u_i = N^{(j)} u_i^{(j)} \tag{64}$$

- Collecting all together, the linear strains and the node displacements are correlated with the following matrix and vector product formulations
- This matrix [*B*] is called the node displacement-strain matrix, or simply [*B*] matrix. *n* represents the number of the nodes found in the single element.

$$\{\varepsilon(\boldsymbol{u})\} = [B]\{u_i^{(n)}\} \tag{65}$$

- *{u*(*n*)*i }* is defined by the following equation.

$$\{u_i^{(n)}\} = \left\{u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, \cdots, u_1^{(n)}, u_2^{(n)}, u_3^{(n)}\right\}^T \tag{66}$$

Node Displacement-Strain Matrix($[B]$ Matrix) 2

- Since $\frac{\partial u_i}{\partial X_j}$ needed in the calculation of the strain represents the quantity of which the node displacement does not depend on the position vector ***x***, we can write as,

$$\frac{\partial u_i}{\partial X_j} = \frac{\partial N^{(n)}}{\partial X_j} u_i^{(n)} \tag{67}$$

- Moreover,

$$\varepsilon_{11} = \frac{\partial u_1}{\partial X_1} \tag{68}$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial X_2} \tag{69}$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial X_3} \tag{70}$$

$$2\varepsilon_{12} = \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \tag{71}$$

$$2\varepsilon_{23} = \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} \tag{72}$$

$$2\varepsilon_{31} = \frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_3} \tag{73}$$

In considering the above,

Node Displacement-Strain Matrix ($[B]$ Matrix) 3

- Specifically, the components are,

$$
\begin{aligned}
\varepsilon_{11} &= \frac{\partial N^{(1)}}{\partial X_1} u_1^{(1)} + \frac{\partial N^{(2)}}{\partial X_1} u_1^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_1} u_1^{(n)} \\
\varepsilon_{22} &= \frac{\partial N^{(1)}}{\partial X_2} u_2^{(1)} + \frac{\partial N^{(2)}}{\partial X_2} u_2^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_2} u_2^{(n)} \\
\varepsilon_{33} &= \frac{\partial N^{(1)}}{\partial X_3} u_3^{(1)} + \frac{\partial N^{(2)}}{\partial X_3} u_3^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_3} u_3^{(n)} \\
2\varepsilon_{12} &= \frac{\partial N^{(1)}}{\partial X_1} u_2^{(1)} + \frac{\partial N^{(2)}}{\partial X_1} u_2^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_1} u_2^{(n)} \\
&\quad + \frac{\partial N^{(1)}}{\partial X_2} u_1^{(1)} + \frac{\partial N^{(2)}}{\partial X_2} u_1^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_2} u_1^{(n)} \\
2\varepsilon_{23} &= \frac{\partial N^{(1)}}{\partial X_2} u_3^{(1)} + \frac{\partial N^{(2)}}{\partial X_2} u_3^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_2} u_3^{(n)} \\
&\quad + \frac{\partial N^{(1)}}{\partial X_3} u_2^{(1)} + \frac{\partial N^{(2)}}{\partial X_3} u_2^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_3} u_2^{(n)} \\
2\varepsilon_{31} &= \frac{\partial N^{(1)}}{\partial X_3} u_1^{(1)} + \frac{\partial N^{(2)}}{\partial X_3} u_1^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_3} u_1^{(n)} \\
&\quad + \frac{\partial N^{(1)}}{\partial X_1} u_3^{(1)} + \frac{\partial N^{(2)}}{\partial X_1} u_3^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_1} u_3^{(n)}
\end{aligned}
\tag{74}
$$

- Based on the components studied in the previous, $[B]$ matrix can be represented in the 6×3 submatrix $[B(k)]$.

$$[B^{(k)}] = \begin{bmatrix} \frac{\partial N^{(k)}}{\partial X_1} & & \\ & \frac{\partial N^{(k)}}{\partial X_2} & \\ & & \frac{\partial N^{(k)}}{\partial X_3} \\ \frac{\partial N^{(k)}}{\partial X_2} & \frac{\partial N^{(k)}}{\partial X_1} & \\ & \frac{\partial N^{(k)}}{\partial X_3} & \frac{\partial N^{(k)}}{\partial X_2} \\ \frac{\partial N^{(k)}}{\partial X_3} & & \frac{\partial N^{(k)}}{\partial X_1} \end{bmatrix} \tag{75}$$

$$[B] = [[B^{(1)}], [B^{(2)}], \ldots, [B^{(n)}]] \tag{76}$$

Element Stiffness Matrix

- By using $[B]$, the integrands $Tij(\boldsymbol{u})$ $\varepsilon ij(\delta\boldsymbol{u})$ found in the first term in $[Ve]$ may be expressed by,

$$\begin{aligned} T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u}) &= \{\varepsilon(\delta\boldsymbol{u})\}^T\,\{\boldsymbol{T}(\boldsymbol{u})\} = \{\varepsilon(\delta\boldsymbol{u})\}^T\,[D]\,\{\varepsilon(\boldsymbol{u})\} \\ &= \{\delta u_i^{(n)}\}^T\,[B]^T\,[D]\,[B]\,\{u_i^{(n)}\} \end{aligned} \tag{77}$$

- $\{\delta u_i^{(n)}\}$, $\{u_i^{(n)}\}$ are the values at the nodal points, and which do not depend on the regional integration because they become constant under the region, thus, we may take them out from the integrals.

$$\begin{aligned} &\int_{\Omega_e} T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u})\,\mathrm{d}\Omega = \int_{\Omega_e} \{\delta u_i^{(n)}\}^T\,[B]^T\,[D]\,[B]\,\{u_i^{(n)}\}\,\mathrm{d}\Omega \\ &= \{\delta u_i^{(n)}\}^T\left[\int_{\Omega_e}[B]^T\,[D]\,[B]\,\mathrm{d}\Omega\right]\{u_i^{(n)}\} \end{aligned} \tag{78}$$

- This integrated matrix is called the element stiffness matrix.

$$[K^{(e)}] = \int_{\Omega_e}[B]^T\,[D]\,[B]\,\mathrm{d}\Omega \tag{79}$$

External Force Vector

- For the second and third terms in the left hand side $[Ve]$, we prepare for the vectors in the node displacements to have them singled out from the integrals.

$$\begin{aligned}\int_{\partial\Omega_e} \delta u_i \cdot t_i \, \mathrm{d}S &= \int_{\partial\Omega_e} \{\delta u_i^{(n)}\}^T [N]^T \{t\} \, \mathrm{d}S \\ &= \{\delta u_i^{(n)}\}^T \int_{\partial\Omega_e} [N]^T \{t\} \, \mathrm{d}S \end{aligned} \tag{80}$$

$$\begin{aligned}\int_{\Omega_e} \rho \, \delta u_i \cdot g_i \, \mathrm{d}\Omega &= \int_{\Omega_e} \rho \, \{\delta u_i^{(n)}\}^T [N]^T \{g\} \, \mathrm{d}\Omega \\ &= \{\delta u_i^{(n)}\}^T \{F^{(e)}\} \int_{\Omega_e} [N]^T \{g\} \, \mathrm{d}\Omega \end{aligned} \tag{81}$$

- Provided that,

$$[N] = \begin{bmatrix} N^{(1)} & & & N^{(2)} & & & & N^{(n)} & & \\ & N^{(1)} & & & N^{(2)} & & \cdots & & N^{(n)} & \\ & & N^{(1)} & & & N^{(2)} & & & & N^{(n)} \end{bmatrix} \tag{82}$$

$$\{t\} = \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix}, \quad \{g\} = \begin{Bmatrix} g_1 \\ g_2 \\ g_3 \end{Bmatrix} \tag{83}$$

- Based on above, the external force vector $\{F^{(e)}\}$ is defined as following,

$$\{F^{(e)}\} = \int_{\partial\Omega_e} [N]^T \{t\} \, \mathrm{d}S + \int_{\Omega_e} \rho \, [N]^T \{g\} \, \mathrm{d}\Omega \tag{84}$$

Total Stiffness Matrix 1

- To put in order,

$$\sum_e \left[\int_{\Omega_e} T_{ij}(\boldsymbol{u})\, \varepsilon_{ij}(\delta \boldsymbol{u})\, \mathrm{d}\Omega - \int_{\partial\Omega_e} \delta \boldsymbol{u} \cdot \boldsymbol{t}\, \mathrm{d}S - \int_{\Omega_e} \rho\, \delta \boldsymbol{u} \cdot \boldsymbol{g}\, \mathrm{d}\Omega \right] = 0 \qquad \forall \delta \boldsymbol{u} \in \mathcal{V} \tag{85}$$

Which can be modified by,

$$\sum_e \left[\{\delta u_i^{(n)}\}^T \left([K^{(e)}]\, \{u_i^{(n)}\} - \{F^{(e)}\} \right) \right] = 0 \tag{86}$$

- Without touching the left hand side, modify $\{\delta u_i^{(n)}\}$, $\{u_i^{(n)}\}$ to the forms, in which the nodal point numbers are provided out of the total numbers instead by the numbers of each element.

$$\{\delta u_1, \delta u_2, \cdots, \delta u_n\} \begin{bmatrix} K_{11} & K_{12} & \ldots & K_{1n} \\ K_{21} & & & \vdots \\ \vdots & & & \vdots \\ K_{n1} & & \ldots & K_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = \{\delta u_1, \delta u_2, \cdots, \delta u_n\} \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix} \tag{87}$$

Unifying the both equations then yield the following,

$$\{\delta u_1, \delta u_2, \cdots, \delta u_n\} \left\{ \begin{bmatrix} K_{11} & K_{12} & \ldots & K_{1n} \\ K_{21} & & & \vdots \\ \vdots & & & \vdots \\ K_{n1} & & \ldots & K_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix} \right\} = \{0\} \tag{88}$$

Total Stiffness Matrix 2

- In order for the equation to form with the arbitrary $\delta \boldsymbol{u}$,

$$\{\delta u_1, \delta u_2, \cdots, \delta u_n\} \left(\begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & & & \vdots \\ \vdots & & & \vdots \\ K_{n1} & & \dots & K_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix} \right) = \{0\} \qquad (89)$$

- The following equation must be established.

$$\begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & & & \vdots \\ \vdots & & & \vdots \\ K_{n1} & & \dots & K_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix} = \{0\} \qquad (90)$$

- Thus, the solutions obtained from the following system of linear equations should be the approximate solutions.

$$\begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & & & \vdots \\ \vdots & & & \vdots \\ K_{n1} & & \dots & K_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix} \qquad (91)$$

- In contructual analysis, this equations are often called the stiffness equations, and its matrix is called the total stiffness matrix.

2-D Finite Element Formulation 1

- In conducting the constructual analysis, there are cases, in which we can simplify the problems under certain conditions.
- The plane strain problem is associated with a situation, where there is a very lengthy wall found in the figure, which being loaded with a load that is unified and perpendicular to the stretch. In this situation, if a middle section in the direction of the stretch is taken out, the displacement in X_2 direction can be observed to be 0. The differentials in X_2 direction is found as 0. Thus, among the 9 components of the strain, $\varepsilon_{22}, \varepsilon_{12}, \varepsilon_{21}, \varepsilon_{23}, \varepsilon_{32}$ of the nine become 0. To make a model with the finite element, the total nodal points in X_2 direction displacements become 0.

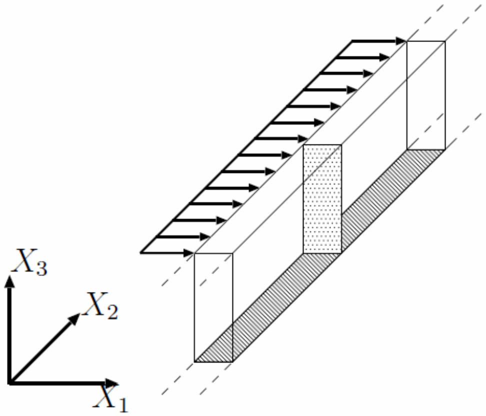

図 3: 平面ひずみ問題の例

2-D Finite Element Formulation 2

- The plane stress problem involves with a situation, in which the board is thin enough to have the board thickness direction 0 with the external shearing stress 0. Thus in the figure, $T_{22}, T_{12}, T_{21}, T_{23}, T_{32}$ become 0.

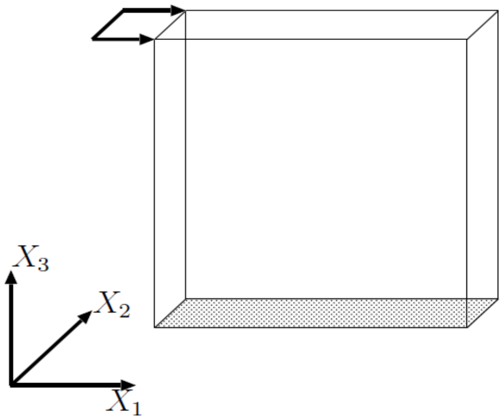

図 4: 平面応力問題の例

- For both, the plane strain problems and the plane stress problems, we can take advantage of these facts to make the calculation easier.

$$\int_{\Omega} T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u})\,\mathrm{d}\Omega = \int_{\partial\Omega} \delta\boldsymbol{u}\cdot\boldsymbol{t}\,\mathrm{d}S + \int_{\Omega} \rho\,\delta\boldsymbol{u}\cdot\boldsymbol{g}\,\mathrm{d}\Omega \qquad \forall\delta\boldsymbol{u}\in\mathcal{V} \qquad (92)$$

Stress-Strain Matrix($[D]$ Matrix) — Plane Strain Problem 1

- First, in the plane strain problems, if the integrand $Tij(\boldsymbol{u})\ \varepsilon(\delta \boldsymbol{u})$ on the left hand side in the first terr $[V_e]$ is not using the summation convention, then expressed by,

$$\begin{aligned} T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u}) = \quad & T_{11}(\boldsymbol{u})\,\varepsilon_{11}(\delta\boldsymbol{u}) + T_{12}(\boldsymbol{u})\,\varepsilon_{12}(\delta\boldsymbol{u}) + T_{13}(\boldsymbol{u})\,\varepsilon_{13}(\delta\boldsymbol{u}) \\ & + T_{21}(\boldsymbol{u})\,\varepsilon_{11}(\delta\boldsymbol{u}) + T_{22}(\boldsymbol{u})\,\varepsilon_{22}(\delta\boldsymbol{u}) + T_{23}(\boldsymbol{u})\,\varepsilon_{23}(\delta\boldsymbol{u}) \\ & + T_{31}(\boldsymbol{u})\,\varepsilon_{31}(\delta\boldsymbol{u}) + T_{32}(\boldsymbol{u})\,\varepsilon_{32}(\delta\boldsymbol{u}) + T_{33}(\boldsymbol{u})\,\varepsilon_{33}(\delta\boldsymbol{u}) \end{aligned} \tag{93}$$

- Here, we suppose the displacement in X_3 direction, and the differentials in X_3 direction to be 0, then $\varepsilon_{33} = \varepsilon_{31} = \varepsilon_{13} = \varepsilon_{23} = \varepsilon_{32} = 0$.

$$T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u}) = \quad T_{11}(\boldsymbol{u})\,\varepsilon_{11}(\delta\boldsymbol{u}) + T_{12}(\boldsymbol{u})\,\varepsilon_{12}(\delta\boldsymbol{u}) + T_{21}(\boldsymbol{u})\,\varepsilon_{11}(\delta\boldsymbol{u}) + T_{22}(\boldsymbol{u})\,\varepsilon_{22}(\delta\boldsymbol{u}) \tag{94}$$

- Using the symmetry property of , $Tij(\boldsymbol{u})$ and $\varepsilon\, ij(\delta \boldsymbol{u})$ about I and j,

$$T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u}) = T_{11}(\boldsymbol{u})\,\varepsilon_{11}(\delta\boldsymbol{u}) + T_{22}(\boldsymbol{u})\,\varepsilon_{22}(\delta\boldsymbol{u}) + 2T_{12}(\boldsymbol{u})\,\varepsilon_{12}(\delta\boldsymbol{u}) = \{\varepsilon(\delta\boldsymbol{u})\}^T\{T(\boldsymbol{u})\} \tag{95}$$

$\{\varepsilon(\boldsymbol{v})\}$, $\{T(\boldsymbol{v})\}$ is defined by the following equations

$$\{\varepsilon(\boldsymbol{v})\} = \begin{Bmatrix} \varepsilon_{11}(\boldsymbol{v}) \\ \varepsilon_{22}(\boldsymbol{v}) \\ 2\varepsilon_{12}(\boldsymbol{v}) \end{Bmatrix} \quad \{T(\boldsymbol{v})\} = \begin{Bmatrix} T_{11}(\boldsymbol{v}) \\ T_{22}(\boldsymbol{v}) \\ T_{12}(\boldsymbol{v}) \end{Bmatrix} \tag{96}$$

Stress-Strain Matrix ($[D]$ Matrix) — Plane Strain Problem 1

- Based on the relationship between stress Tij and the strain εij, correlate $\{T(\boldsymbol{u})\}$ *and* $\{\varepsilon(\boldsymbol{u})\}$ with the matrix and vector product formulations,

$$\{T(\boldsymbol{u})\} = [D]\{\varepsilon(\boldsymbol{u})\} \tag{97}$$

- Components of Tij in $\{T(\boldsymbol{u})\}$ are given by,

$$\begin{aligned} T_{11} &= \kappa(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2G\varepsilon_{11} - \frac{2G}{3}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \\ T_{22} &= \kappa(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) + 2G\varepsilon_{22} - \frac{2G}{3}(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) \\ T_{12} &= 2G\varepsilon_{12} \end{aligned} \tag{98}$$

- Here substitute $\varepsilon_{33} = \varepsilon_{31} = \varepsilon_{13} = \varepsilon_{23} = \varepsilon_{32} = 0$.

$$\begin{aligned} T_{11} &= \kappa(\varepsilon_{11} + \varepsilon_{22}) + 2G\varepsilon_{11} - \frac{2G}{3}(\varepsilon_{11} + \varepsilon_{22}) \\ T_{22} &= \kappa(\varepsilon_{11} + \varepsilon_{22}) + 2G\varepsilon_{22} - \frac{2G}{3}(\varepsilon_{11} + \varepsilon_{22}) \\ T_{12} &= 2G\varepsilon_{12} \end{aligned} \tag{99}$$

Stress-Strain Matrix ($[D]$ Matrix) — Plane Strain Problems 2

- Effectively, put them into the matrix notation,

$$\begin{Bmatrix} T_{11} \\ T_{22} \\ T_{12} \end{Bmatrix} = \begin{bmatrix} \kappa & \kappa & 0 \\ \kappa & \kappa & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} + \begin{bmatrix} \frac{4G}{3} & -\frac{2G}{3} & 0 \\ -\frac{2G}{3} & \frac{4G}{3} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} \tag{100}$$

In concerning with the development of our discussions in later lectures, here we define $[D_v]$, $[D_d]$

$$[D_v] = \begin{bmatrix} \kappa & \kappa & 0 \\ \kappa & \kappa & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [D_d] = \begin{bmatrix} \frac{4G}{3} & -\frac{2G}{3} & 0 \\ -\frac{2G}{3} & \frac{4G}{3} & 0 \\ 0 & 0 & G \end{bmatrix} \tag{101}$$

- Defining $[D]$, then we can make a correlation in the following form.

$$[D] = [D_v] + [D_d] \tag{102}$$

- Integrand $Tij(\boldsymbol{u})$ $\varepsilon ij(\delta \boldsymbol{u})$ on the left hand side in the first term can be expressed by,

$$T_{ij}(\boldsymbol{u})\, \varepsilon_{ij}(\delta \boldsymbol{u}) = \{\varepsilon(\delta \boldsymbol{u})\}^T [D] \{\varepsilon(\boldsymbol{u})\} \tag{103}$$

Stress-Strain Matrix($[D]$ Matrix) — Plane Stress Problems 1

- In the plane stress, the integrand $Tij(\boldsymbol{u})\ \varepsilon ij(\delta \boldsymbol{u})$ on the left hand side the first term in $[V_e]$ can be expressed without using the summation convention,.

$$\begin{aligned} T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u}) = \quad & T_{11}(\boldsymbol{u})\,\varepsilon_{11}(\delta\boldsymbol{u}) + T_{12}(\boldsymbol{u})\,\varepsilon_{12}(\delta\boldsymbol{u}) + T_{13}(\boldsymbol{u})\,\varepsilon_{13}(\delta\boldsymbol{u}) \\ & + T_{21}(\boldsymbol{u})\,\varepsilon_{11}(\delta\boldsymbol{u}) + T_{22}(\boldsymbol{u})\,\varepsilon_{22}(\delta\boldsymbol{u}) + T_{23}(\boldsymbol{u})\,\varepsilon_{23}(\delta\boldsymbol{u}) \\ & + T_{31}(\boldsymbol{u})\,\varepsilon_{31}(\delta\boldsymbol{u}) + T_{32}(\boldsymbol{u})\,\varepsilon_{32}(\delta\boldsymbol{u}) + T_{33}(\boldsymbol{u})\,\varepsilon_{33}(\delta\boldsymbol{u}) \end{aligned} \tag{104}$$

- Here suppose the plane stress in x_3 direction, then $T_{33} = T_{31} = T_{13} = T_{23} = T_{32} = 0$. Thus we can simplify,

$$T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u}) = \quad T_{11}(\boldsymbol{u})\,\varepsilon_{11}(\delta\boldsymbol{u}) + T_{12}(\boldsymbol{u})\,\varepsilon_{12}(\delta\boldsymbol{u}) + T_{21}(\boldsymbol{u})\,\varepsilon_{11}(\delta\boldsymbol{u}) + T_{22}(\boldsymbol{u})\,\varepsilon_{22}(\delta\boldsymbol{u}) \tag{105}$$

- Using the symmetry property of , $Tij(\boldsymbol{u})$ and $\varepsilon ij(\delta \boldsymbol{u})$ about I and j,

$$\begin{aligned} T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u}) =& T_{11}(\boldsymbol{u})\,\varepsilon_{11}(\delta\boldsymbol{u}) + T_{22}(\boldsymbol{u})\,\varepsilon_{22}(\delta\boldsymbol{u}) + 2T_{12}(\boldsymbol{u})\,\varepsilon_{12}(\delta\boldsymbol{u}) \\ =& \{\varepsilon(\delta\boldsymbol{u})\}^T\{T(\boldsymbol{u})\} \end{aligned} \tag{106}$$

$\{\varepsilon(\boldsymbol{v})\}$, $\{T(\boldsymbol{v})\}$ are defined by the following equations.

$$\{\varepsilon(\boldsymbol{v})\} = \begin{Bmatrix} \varepsilon_{11}(\boldsymbol{v}) \\ \varepsilon_{22}(\boldsymbol{v}) \\ 2\varepsilon_{12}(\boldsymbol{v}) \end{Bmatrix} \quad \{T(\boldsymbol{v})\} = \begin{Bmatrix} T_{11}(\boldsymbol{v}) \\ T_{22}(\boldsymbol{v}) \\ T_{12}(\boldsymbol{v}) \end{Bmatrix} \tag{107}$$

Stress-Strain Matrix($[D]$ Matrix) — Plane Stress Problems 2

- Based on the relations between the stress T_{ij} and strain ε_{ij},we can correlate $\{T(\boldsymbol{u})\}$ and $\{\varepsilon(\boldsymbol{u})\}$ with the matrix and the vector product forms.

$$\{T(\boldsymbol{u})\} = [D]\{\varepsilon(\boldsymbol{u})\} \tag{108}$$

- Components of T_{ij} in $\{T(\boldsymbol{u})\}$ are given by,

$$\begin{aligned}
T_{11} &= \kappa\left(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}\right)+2G\varepsilon_{11}-\frac{2G}{3}\left(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}\right)\\
T_{22} &= \kappa\left(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}\right)+2G\varepsilon_{22}-\frac{2G}{3}\left(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}\right)\\
T_{33} &= \kappa\left(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}\right)+2G\varepsilon_{33}-\frac{2G}{3}\left(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}\right)\\
T_{12} &= 2G\varepsilon_{12}\\
T_{23} &= 2G\varepsilon_{23}\\
T_{31} &= 2G\varepsilon_{31}
\end{aligned} \tag{109}$$

- Using $T_{33} = T_{23} = T_{31} = 0$,

$$\kappa\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) + 2G\varepsilon_{33} - \frac{2G}{3}\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) = 0 \tag{110}$$

$$\varepsilon_{23} = 0 \tag{111}$$

$$\varepsilon_{31} = 0 \tag{112}$$

- If we conduct further transformation, the following relation may be given.

$$\varepsilon_{33} = \frac{\frac{2G}{3} - \kappa}{\frac{4G}{3} + \kappa}\left(\varepsilon_{11} + \varepsilon_{22}\right) \tag{113}$$

Stress-Strain Matrix($[D]$ Matrix) — Plane Stress Problems 3

- Then we obtain the following relations.

$$
\begin{aligned}
T_{11} &= \frac{2G\left(\dfrac{2G}{3}+2\kappa\right)}{\dfrac{4G}{3}+\kappa}\varepsilon_{11} + \frac{2G\left(\kappa-\dfrac{2G}{3}\right)}{\dfrac{4G}{3}+\kappa}\varepsilon_{22} \\
T_{22} &= \frac{2G\left(\kappa-\dfrac{2G}{3}\right)}{\dfrac{4G}{3}+\kappa}\varepsilon_{11} + \frac{2G\left(\dfrac{2G}{3}+2\kappa\right)}{\dfrac{4G}{3}+\kappa}\varepsilon_{22}
\end{aligned}
\tag{114}
$$

- Bring then into the matrix representations, then given by the following

$$
\begin{Bmatrix} T_{11} \\ T_{22} \\ T_{12} \end{Bmatrix} = \begin{bmatrix} \dfrac{2G\left(\dfrac{2G}{3}+2\kappa\right)}{\dfrac{4G}{3}+\kappa} & \dfrac{2G\left(\kappa-\dfrac{2G}{3}\right)}{\dfrac{4G}{3}+\kappa} & 0 \\ \dfrac{2G\left(\kappa-\dfrac{2G}{3}\right)}{\dfrac{4G}{3}+\kappa} & \dfrac{2G\left(\dfrac{2G}{3}+2\kappa\right)}{\dfrac{4G}{3}+\kappa} & 0 \\ 0 & 0 & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix}
\tag{115}
$$

Stress-Strain Matrix($[D]$ Matrix) — Plane Stress Problems 4

- We can express the integrand $Tij(\boldsymbol{u})\ \varepsilon ij(\ \delta\boldsymbol{u})$ on the left hand side in the first term by,

$$[D] = \begin{bmatrix} \dfrac{2G\left(\dfrac{2G}{3} + 2\kappa\right)}{\dfrac{4G}{3} + \kappa} & \dfrac{2G\left(\kappa - \dfrac{2G}{3}\right)}{\dfrac{4G}{3} + \kappa} & 0 \\ \dfrac{2G\left(\kappa - \dfrac{2G}{3}\right)}{\dfrac{4G}{3} + \kappa} & \dfrac{2G\left(\dfrac{2G}{3} + 2\kappa\right)}{\dfrac{4G}{3} + \kappa} & 0 \\ 0 & 0 & G \end{bmatrix} \tag{116}$$

$$T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u}) = \{\varepsilon(\delta\boldsymbol{u})\}^T[D]\,\{\varepsilon(\boldsymbol{u})\} \tag{117}$$

Node Displacement-Strain Matrix ($[B]$ Matrix) 1

- Displacement and the linear strain

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \tag{118}$$

- Displacement and the node displacement

$$u_i = N^{(j)} u_i^{(j)} \tag{119}$$

- When we assume a plane strain or a plane stress, the strain vector representations $\{\varepsilon(\boldsymbol{v})\}$ includes only the components $\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}$ thus the displacement in X_3 direction is not being used.
- All together, the linear strain and the node displacement are correlated with the matrix and vector product forms.

$$\{\varepsilon(\boldsymbol{u})\} = [B]\,\{u_i^{(n)}\} \tag{120}$$

$\{u_i^{(n)}\}$ is defined by the following equations.

$$\{u_i^{(n)}\} = \begin{Bmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_1^{(2)} \\ u_2^{(2)} \\ \vdots \\ u_1^{(n)} \\ u_2^{(n)} \end{Bmatrix} \tag{121}$$

Node Displacement-Strain Matrix ($[B]$ Matrix) 1

- Since $\frac{\partial u_i}{\partial X_j}$ needed in the calculation of strain is the quantity of which the node

displacement does not depend on the position vector ***x***, we can write as,

$$\frac{\partial u_i}{\partial X_j} = \frac{\partial N^{(n)}}{\partial X_j} u_i^{(n)} \tag{122}$$

- Moreover,

$$\varepsilon_{11} = \frac{\partial u_1}{\partial X_1} \tag{123}$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial X_2} \tag{124}$$

$$2\varepsilon_{12} = \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \tag{125}$$

Node Displacement-Strain Matrix ($[B]$ Matrix) 2

- Components are,

$$\begin{aligned}
\varepsilon_{11} &= \frac{\partial N^{(1)}}{\partial X_1} u_1^{(1)} + \frac{\partial N^{(2)}}{\partial X_1} u_1^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_1} u_1^{(n)} \\
\varepsilon_{22} &= \frac{\partial N^{(1)}}{\partial X_2} u_2^{(1)} + \frac{\partial N^{(2)}}{\partial X_2} u_2^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_2} u_2^{(n)} \\
2\varepsilon_{12} &= \frac{\partial N^{(1)}}{\partial X_1} u_2^{(1)} + \frac{\partial N^{(2)}}{\partial X_1} u_2^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_1} u_2^{(n)} \\
&\quad + \frac{\partial N^{(1)}}{\partial X_2} u_1^{(1)} + \frac{\partial N^{(2)}}{\partial X_2} u_1^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_2} u_1^{(n)}
\end{aligned} \tag{126}$$

- Based on this, $[B]$ matrix can be represented in the 3×2 submatrix $[B^{(k)}]$.

$$[B^{(k)}] = \begin{bmatrix} \frac{\partial N^{(k)}}{\partial X_1} & \\ & \frac{\partial N^{(k)}}{\partial X_2} \\ \frac{\partial N^{(k)}}{\partial X_2} & \frac{\partial N^{(k)}}{\partial X_1} \end{bmatrix} \tag{127}$$

$$[B] = \Big[[B^{(1)}], [B^{(2)}], \ldots, [B^{(n)}] \Big] \tag{128}$$

Element Stiffness Matrix

- Using $[B]$, integrand $T_{ij}(\boldsymbol{u})\ \varepsilon_{ij}(\delta\boldsymbol{u})$ in the first term in $[V_e]$ can be expressed as,

$$\begin{aligned} T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u}) &= \{\varepsilon(\delta\boldsymbol{u})\}^T\,\{\boldsymbol{T}(\boldsymbol{u})\} \\ &= \{\varepsilon(\delta\boldsymbol{u})\}^T\,[D]\,\{\varepsilon(\boldsymbol{u})\} \\ &= \{\delta u_i^{(n)}\}^T\,[B]^T\,[D]\,[B]\,\{u_i^{(n)}\} \end{aligned} \qquad (129)$$

- $\{\delta u_i^{(n)}\}$, $\{u_i^{(n)}\}$ are the values at the nodal points, and which do not depend on the regional integration because they become constant under the region, therefore, we may take them out from the integrals.

$$\begin{aligned} &\int_{\Omega_e} \{\delta u_i^{(n)}\}^T\,[B]^T\,[D]\,[B]\,\{u_i^{(n)}\}\,\mathrm{d}\Omega \\ &= \{\delta u_i^{(n)}\}^T \left[\int_{\Omega_e} [B]^T\,[D]\,[B]\,\mathrm{d}\Omega\right] \{u_i^{(n)}\} \end{aligned} \qquad (130)$$

- This matrix is called the element stiffness matrix

$$[K^{(e)}] = \int_{\Omega_e} [B]^T\,[D]\,[B]\,\mathrm{d}\Omega \qquad (131)$$

External Force Vector

- For the second and third terms in the left hand side $[V_e]$, we prepare for the vectors in the node displacements to have them singled out from the integrals.

$$\begin{aligned}\int_{\partial\Omega_e} \delta u_i \cdot t_i \,\mathrm{d}S &= \int_{\partial\Omega_e} \{\delta u_i^{(n)}\}^T \,[N]^T \,\{t\} \,\mathrm{d}S \\ &= \{\delta u_i^{(n)}\}^T \int_{\partial\Omega_e} [N]^T \,\{t\} \,\mathrm{d}S \qquad (132)\end{aligned}$$

$$\begin{aligned}\int_{\Omega_e} \rho \,\delta u_i \cdot g_i \,\mathrm{d}\Omega &= \int_{\Omega_e} \rho \,\{\delta u_i^{(n)}\}^T \,[N]^T \,\{g\} \,\mathrm{d}\Omega \\ &= \{\delta u_i^{(n)}\}^T \int_{\Omega_e} \rho \,[N]^T \,\{g\} \,\mathrm{d}\Omega \qquad (133)\end{aligned}$$

- Provided that,

$$[N] = \begin{bmatrix} N^{(1)} & & N^{(2)} & & & N^{(n)} & \\ & N^{(1)} & & N^{(2)} & \dots & & N^{(n)} \end{bmatrix} \tag{134}$$

$$\{t\} = \begin{Bmatrix} t_1 \\ t_2 \end{Bmatrix} \tag{135}$$

$$\{g\} = \begin{Bmatrix} g_1 \\ g_2 \end{Bmatrix} \tag{136}$$

- The external force vector $\{F^{(e)}\}$ is defined as

$$\{F^{(e)}\} = \int_{\partial\Omega_e} [N]^T \{t\}\, \mathrm{d}S + \int_{\Omega_e} \rho\, [N]^T \{g\}\, \mathrm{d}\Omega \tag{137}$$

- From an element stiffness matrix and the external force vector we obtained, the stiffness equations can be formed in exact the same way we did in 3-D.

δu remains *O* in surface region given by the displacement boundary

Kubo.,Tosiaki,”Fundamental Tensor Analysis(Tensor Kaiseki no Kiso): Review on Virtual Work Principle (Kaso Shigoto no Genri no Fukusyu)”p.181 6.3

- Boundary value problem

[1] Balance equations(Cauchy’s equation of motion)

$$\nabla_x \cdot \boldsymbol{T} + \rho \boldsymbol{g} = \boldsymbol{O} \tag{138}$$

[2] Boundary condition equations

$$\boldsymbol{T}^{\boldsymbol{T}} \cdot \boldsymbol{n} = \underline{\boldsymbol{t}} \tag{139}$$

$$\boldsymbol{u} = \underline{\boldsymbol{u}} \tag{140}$$

[3] Displacement・strain relational expression

[4] Stress strain relational expression（Constructive equation）

- A continuous univalent stress field, which satisfies the balance equations and the stress boundary condition $\boldsymbol{T}^{\boldsymbol{T}} \cdot \boldsymbol{n} = \underline{\boldsymbol{t}}$ is called mechanically admissible.
- A continuous univalent displacement field, which satisfies the displacement boundary condition $\boldsymbol{u} = \underline{\boldsymbol{u}}$ is called geometrically admissible.

- An admissible stress field and the admissible displacement field may be independently assumed.

Suppose we have $\check{\boldsymbol{T}}$ and $\check{\check{\boldsymbol{u}}}$.

$\longrightarrow$ $\check{\boldsymbol{T}}$, from [3] and [4], we assume the displacement field, which gives the stress field $\check{\boldsymbol{T}}$, is being determined as $\check{\boldsymbol{u}}$. Then $\check{\boldsymbol{u}}$ becomes different from $\check{\check{\boldsymbol{u}}}$. (not guaranteed for the complete determination)

Which satisfies the balance equations,

$$\nabla_x \cdot \check{\boldsymbol{T}} + \rho \boldsymbol{g} = \boldsymbol{O} \tag{141}$$

Therefore

$$\int_V (\nabla_x \cdot \check{\boldsymbol{T}} + \rho \boldsymbol{g}) \cdot \check{\boldsymbol{u}}\, dv = 0 \tag{142}$$

Transforming the equations to yield,

$$\int_V \check{\boldsymbol{T}} : (\check{\boldsymbol{u}} \otimes \nabla_x)\ dv = \int_{S_t} \boldsymbol{t} \cdot \check{\boldsymbol{u}}\, ds + \int_{S_u} \boldsymbol{n} \cdot \check{\boldsymbol{T}} \cdot \check{\boldsymbol{u}}\, ds + \int_V \rho \boldsymbol{g} \cdot \check{\boldsymbol{u}}\, dv \tag{143}$$

- We assume $\boldsymbol{T}$ and $\boldsymbol{u}$**,** which satisfies [1]-[4] are gained. The arbitrary $\check{\check{\boldsymbol{u}}}$ can be written as,

$$\check{\boldsymbol{u}} = \boldsymbol{u} + \delta \boldsymbol{u} \tag{144}$$

Provided that $\delta \boldsymbol{u}$ becomes 0 in the surface region given by the displacement boundary condition.

Substitute this into (143).

$$\int_V \boldsymbol{T} : \{(\boldsymbol{u} + \delta\boldsymbol{u}) \otimes \nabla_x\}\, dv = \int_{S_t} \boldsymbol{t} \cdot (\boldsymbol{u} + \delta\boldsymbol{u})\ ds + \int_{S_u} \boldsymbol{n} \cdot \boldsymbol{T} \cdot (\boldsymbol{u} + \delta\boldsymbol{u})\ ds + \int_V \rho\boldsymbol{g} \cdot (\boldsymbol{u} + \delta\boldsymbol{u})\ dv \tag{145}$$

Also,(143) can be formed by defining $\tilde{\tilde{u}}$ to use ***u*,**

$$\int_V \boldsymbol{T} : (\boldsymbol{u} \otimes \nabla_x)\ dv = \int_{S_t} \boldsymbol{t} \cdot \boldsymbol{u}\, ds + \int_{S_u} \boldsymbol{n} \cdot \boldsymbol{T} \cdot \boldsymbol{u}\, ds + \int_V \rho\boldsymbol{g} \cdot \boldsymbol{u}\, dv \tag{146}$$

- Take the difference from both sides in(145)and (146),

$$\int_V \boldsymbol{T} : (\delta\boldsymbol{u} \otimes \nabla_x)\ dv = \int_{S_t} \boldsymbol{t} \cdot \delta\boldsymbol{u}\, ds + \int_{S_u} \boldsymbol{n} \cdot \boldsymbol{T} \cdot \delta\boldsymbol{u}\, ds + \int_V \rho\boldsymbol{g} \cdot \delta\boldsymbol{u}\, dv \tag{147}$$

The virtual work equations are obtained.

2004 Advanced Nonlinear Finite Element Method Exercises 1

- Using the stress strain relational expressions in(10), the third term in (18)can be derived from the second term in the same equation. Write down and derive the third term.
- Verify (30) from (20)
- Verify (40) from(32)
- Verify (79) from (48)
- Show an example of the problem, in which the plane strain and the plane stress can be actually adopted.
- State what can be found in estimating the plane strain and the plane stress, then derive the stress-strain matrix for each.
- Verify (147) from (141)
- Make explanations over the admissible stress field and the admissible displacement field, then state the meaning for the following: the admissible stress field and the admissible displacement field may be independently assumed.