Nonlinear
Finite Element Method

01/11/2004

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- Lectures include discussion of the nonlinear finite element method.
- It is preferable to have completed “Introduction to Nonlinear Finite Element Analysis” available in summer session.
- If not, students are required to study on their own before participating this course. Reference:Toshiaki.,Kubo. “Introduction: Tensor Analysis For Nonlinear Finite Element Method” (Hisennkei Yugen Yoso no tameno Tensor Kaiseki no Kiso),Maruzen.
- Lecture references are available and downloadable at http://www.sml.k.u-tokyo.ac.jp/members/nabe/lecture2004 They should be posted on the website by the day before scheduled meeting, and each students are expected to come in with a copy of the reference.
- Lecture notes from previous year are available and downloadable, also at http://www.sml.k.u.tokyo.ac.jp/members/nabe/lecture2003 You may find the course title, ”Advanced Finite Element Method” but the contents covered are the same I will cover this year.
- I will assign the exercises from this year, and expect the students to hand them in during the following lecture. They are not the requirements and they will not be graded, however it is important to actually practice calculate in deeper understanding the finite element method.
- For any questions, contact me at nabe@sml.k.u-tokyo.ac.jp

Nonlinear Finite Element Method Lecture Schedule

1. 10/ 4 Finite element analysis in boundary value problems and the differential equations
2. 10/18 Finite element analysis in linear elastic body
3. 10/25 Isoparametric solid element (program)
4. 11/ 1 Numerical solution and boundary condition processing for system of linear equations（with exercises）
5. 11/ 8 Basic program structure of the linear finite element method(program)
6. 11/15 Finite element formulation in geometric nonlinear problems(program)
7. 11/22 Static analysis technique、hyperelastic body and elastic-plastic material for nonlinear equations (program)
8. 11/29 Exercises for Lecture7
9. 12/ 6 Dynamic analysis technique and eigenvalue analysis in the nonlinear equations
10. 12/13 Structural element
11. 12/20 Numerical solution— skyline method、iterative method for the system of linear equations
12. 1/17 ALE finite element fluid analysis
13. 1/24 ALE finite element fluid analysis

Examples of Gauss Elimination 1

- Consider the following system of linear equations. We define coefficient matrix as $[A]$, unknown vector as $\{b\}$, and the right hand side as $\{c\}$.

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- **[step 1.]** In order to obtain 0 below the diagonal term for the first row in$[A(1)]$, operate the equation in 2 ～ 4 columns including the right hand side.

$$2\text{ 行目} - \{\underbrace{(-4)}_{A_{21}^{(1)}} / \underbrace{5}_{A_{11}^{(1)}}\} \times 1\text{ 行目}$$

$$3\text{ 行目} - \{\underbrace{1}_{A_{31}^{(1)}} / \underbrace{5}_{A_{11}^{(1)}}\} \times 1\text{ 行目}$$

$$4\text{ 行目} - \{\underbrace{0}_{A_{41}^{(1)}} / \underbrace{5}_{A_{11}^{(1)}}\} \times 1\text{ 行目}$$

- By this operation, $[A^{(1)}]\{b\} = \{c^{(1)}\}$ is transformed as following then express it as $[A^{(2)}]\{b\} = \{c^{(2)}\}$

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & -\frac{16}{5} & \frac{29}{5} & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Forward Elimination 1

- **[step 2.]**

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & -\frac{16}{5} & \frac{29}{5} & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- In order to obtain 0 below the diagonal term for the second row in$[A^{(2)}]$, operate the equation in 3 and 4 columns including the right hand side.

$$3\,行目 - \left\{ \underbrace{\left(-\frac{16}{5}\right)}_{A^{(2)}_{23}} \Big/ \underbrace{\frac{14}{5}}_{A^{(2)}_{22}} \right\} \times 2\,行目$$

$$4\,行目 - \left\{ \underbrace{1}_{A^{(2)}_{24}} \Big/ \underbrace{\frac{14}{5}}_{A^{(2)}_{22}} \right\} \times 2\,行目$$

- By this operation,$[A^{(2)}]\{b\} = \{c^{(2)}\}$ is transformed as in the following then express it as $[A^{(3)}]\{b\} = \{c^{(3)}\}$

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & 0 & \frac{15}{7} & -\frac{20}{7} \\ 0 & 0 & -\frac{20}{7} & \frac{65}{14} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{8}{7} \\ -\frac{5}{14} \end{bmatrix}$$

Forward Elimination 2

- **[step 3.]**

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & 0 & \frac{15}{7} & -\frac{20}{7} \\ 0 & 0 & -\frac{20}{7} & \frac{65}{14} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{8}{7} \\ -\frac{5}{14} \end{bmatrix}$$

- In order to obtain 0 below the diagonal term for the third row in$[A^{(3)}]$, operate the equation in the fourth columns including the right hand side.

$$4\text{ 行目} - \left\{ \underbrace{\left(-\frac{20}{7}\right)}_{A_{34}^{(3)}} \Big/ \underbrace{\frac{15}{7}}_{A_{44}^{(3)}} \right\} \times 3\text{ 行目}$$

By this operation, $[A^{(3)}]\{b\} = \{c^{(3)}\}$ is transformed as in the following then express it as $[A^{(4)}]\{b\} = \{c^{(4)}\}$

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & 0 & \frac{15}{7} & -\frac{20}{7} \\ 0 & 0 & 0 & \frac{5}{6} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{8}{7} \\ \frac{7}{6} \end{bmatrix}$$

- The system of linear equations is transformed to possess upper triangular matrix 1 as coefficient matrix. Such operation is called the forward elimination.

The matrix with its components to be 0 below 1 diagonal term is called the upper triangular matrix. In the same way, for the matrix with its components to be 0 above the diagonal term is called lower triangular matrix.

Backward Substitution

- In this way, the unknown numbers $\{b\}$ of the system of linear equation with its coefficient matrix being triangular matrix, can be obtained by an order b_4, b_3, b_2, b_1. Such operation is called the backward substitution.

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & 0 & \frac{15}{7} & -\frac{20}{7} \\ 0 & 0 & 0 & \frac{5}{6} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{8}{7} \\ \frac{7}{6} \end{bmatrix}$$

[step 1.]

$$b_4 = \underbrace{\frac{7}{6}}_{c_4^{(4)}} \Big/ \underbrace{\frac{5}{6}}_{A_{44}^{(4)}} = \frac{7}{5}$$

[step 2.]

$$b_3 = \left\{ \underbrace{\frac{8}{7}}_{c_3^{(4)}} - \underbrace{\left(-\frac{20}{7}\right)}_{A_{34}^{(4)}} \cdot \underbrace{\frac{7}{5}}_{b_4} \right\} \Big/ \underbrace{\frac{15}{7}}_{A_{33}^{(4)}} = \frac{12}{5}$$

[step 3.]

$$b_2 = \left\{ \underbrace{1}_{c_2^{(4)}} - \underbrace{\left(-\frac{16}{5}\right)}_{A_{23}^{(4)}} \cdot \underbrace{\frac{12}{5}}_{b_3} - \underbrace{1}_{A_{24}^{(4)}} \cdot \underbrace{\frac{7}{5}}_{b_4} \right\} \Big/ \underbrace{\frac{14}{5}}_{A_{22}^{(4)}} = \frac{13}{5}$$

[step 4.]

$$b_1 = \left\{ \underbrace{0}_{c_1^{(4)}} - \underbrace{(-4)}_{A_{12}^{(4)}} \cdot \underbrace{\frac{13}{5}}_{b_2} - \underbrace{1}_{A_{13}^{(4)}} \cdot \underbrace{\frac{12}{5}}_{b_3} - \underbrace{0}_{A_{14}^{(4)}} \cdot \underbrace{\frac{7}{5}}_{b_4} \right\} \Big/ \underbrace{5}_{A_{11}^{(4)}} = \frac{8}{5}$$

Backward Elimination 1

- Or, we can take the following steps, and which is called the backward elimination. Here, we define .

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & 0 & \frac{15}{7} & -\frac{20}{7} \\ 0 & 0 & 0 & \frac{5}{6} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{8}{7} \\ \frac{7}{6} \end{bmatrix}$$

[step 1.]

First, b_4 can be obtained by the equation in the fourth column.

$$b_4 = \underbrace{\frac{7}{6}}_{c_4^{(4,0)}} \Big/ \underbrace{\frac{5}{6}}_{A_{44}^{(4)}} = \frac{7}{5}$$

Now, get back to the initial system of linear equations,

$$\begin{cases} 5b_1 & + -4b_2 & + \ 1b_3 & + \ 0b_4 & = 0 \\ 0b_1 & + \ \frac{14}{5}b_2 & + -\frac{16}{5}b_3 & + \ 1b_4 & = 1 \\ 0b_1 & + \ 0b_2 & + \ \frac{15}{7}b_3 & + -\frac{20}{7}b_4 & = \frac{8}{7} \\ 0b_1 & + \ 0b_2 & + \ 0b_3 & + \ \frac{5}{6}b_4 & = \frac{7}{6} \end{cases}$$

The fourth column can be ignored since we already obtained b_4.

For the column $1 \sim 3$, b_4 is already evaluated thus transpositioned to the left hand side.

$$\begin{cases} 5b_1 & + -4b_2 & + \quad 1b_3 & = 0 & -(\quad 0b_4) \\ 0b_1 & + \quad \frac{14}{5}b_2 & + -\frac{16}{5}b_3 & = 1 & -(\quad 1b_4) \\ 0b_1 & + \quad 0b_2 & + \quad \frac{15}{7}b_3 & = \frac{8}{7} & -(-\frac{20}{7}b_4) \end{cases}$$

We can express above into matrix notation,

$$\begin{bmatrix} 5 & -4 & 1 \\ 0 & \frac{14}{5} & -\frac{16}{5} \\ 0 & 0 & \frac{15}{7} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{8}{7} \end{bmatrix} - b_4 \begin{bmatrix} 0 \\ 1 \\ -\frac{20}{7} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{5} \\ \frac{36}{7} \end{bmatrix}$$

Take the values on the right hand side in consideration, we can find the following treatment being conducted.

$$c_1^{(4,1)} = \underbrace{0}_{c_1^{(4,0)}} - \underbrace{0}_{A_{14}^{(4)}} \cdot \underbrace{\frac{7}{5}}_{b_4} = 0$$

$$c_2^{(4,1)} = \underbrace{1}_{c_2^{(4,0)}} - \underbrace{1}_{A_{24}^{(4)}} \cdot \underbrace{\frac{7}{5}}_{b_4} = -\frac{2}{5}$$

$$c_3^{(4,1)} = \underbrace{\frac{8}{7}}_{c_3^{(4,0)}} - \underbrace{\left(-\frac{20}{7}\right)}_{A_{34}^{(4)}} \cdot \underbrace{\frac{7}{5}}_{b_4} = \frac{36}{7}$$

Backward Elimination 2

$$\begin{bmatrix} 5 & -4 & 1 \\ 0 & \frac{14}{5} & -\frac{16}{5} \\ 0 & 0 & \frac{15}{7} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{5} \\ \frac{36}{7} \end{bmatrix}$$

[step 2.]

$$b_3 = \underbrace{\frac{36}{7}}_{c_3^{(4,1)}} \Big/ \underbrace{\frac{15}{7}}_{A_{33}^{(4)}} = \frac{12}{5}$$

$$c_1^{(4,2)} = \underbrace{0}_{c_1^{(4,1)}} - \underbrace{1}_{A_{13}^{(4)}} \cdot \underbrace{\frac{12}{5}}_{b_3} = -\frac{12}{5}$$

$$c_2^{(4,2)} = \underbrace{-\frac{2}{5}}_{c_2^{(4,1)}} - \underbrace{\left(-\frac{16}{5}\right)}_{A_{23}^{(4)}} \cdot \underbrace{\frac{12}{5}}_{b_3} = \frac{182}{25}$$

$$\begin{bmatrix} 5 & -4 \\ 0 & \frac{14}{5} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{2}{5} \end{bmatrix} - b_3 \begin{bmatrix} 1 \\ -\frac{16}{5} \end{bmatrix} = \begin{bmatrix} -\frac{12}{5} \\ \frac{182}{25} \end{bmatrix}$$

[step 3.]

$$b_2 = \underbrace{\frac{182}{25}}_{c_2^{(4,2)}} \bigg/ \underbrace{\frac{14}{5}}_{A_{22}^{(4)}} = \frac{13}{5}$$

$$c_1^{(4,3)} = \underbrace{-\frac{12}{5}}_{c_1^{(4,2)}} - \underbrace{(-4)}_{A_{12}^{(4)}} \cdot \underbrace{\frac{13}{5}}_{b_2} = 8$$

$$5 \cdot b_1 = -\frac{12}{5} - b_2 \cdot (-4) = 8$$

[step 4.]

$$b_1 = \underbrace{8}_{c_1^{(4,3)}} \bigg/ \underbrace{5}_{A_{11}^{(4)}} = \frac{8}{5}$$

Backward Substitution and Backward Elimination

- As we can see, the backward substitution and the backward elimination take the procedures so unlike to each other. The backward elimination appears to be more complex in hand calculation, though the operand needed in obtaining the unknown numbers stays in common for the two operations.
- To take a notice of coefficient matrix components appear in the steps, columns are fixed in the backward substitution, while rows are fixed in the backward substitution.
- The difference between the backward substitution and the backward elimination can be found in the coding process where assignment of loop index is reflected by the two operations, and the two operations :backward substitution and backward elimination are properly picked by the store method of coefficient matrix.
- In actual finite element coding, skyline method, an efficiency promoted technique based on Gauss elimination, is commonly used as a standard, yet the skyline method adopts the backward elimination.

Triangular Factorization 1

- The most characteristics of a series of operation implemented in Gauss elimination is that in transforming the coefficient matrix into the upper triangular matrix, the values in the right hand side is not being employed.
- Thus, we can implement the transformation into upper triangular matrix only by values of coefficient matrix.
- We will express the steps by elementary matrix product.
- Elementary matrix $[P(i, j:C)]$ is consisted of the diagonal component 1, i, j component($i \neq j$) of C, and the rest 0.

$$[P(i,j:C)] = \begin{array}{cc} & \begin{array}{ccccccc} & & j列 & & & & \end{array} \\ \begin{array}{c} \\ \\ \\ \\ i行 \\ \\ \\ \end{array} & \begin{bmatrix} 1 & & \vdots & & & & 0 \\ & 1 & \vdots & & & & \\ & & 1 & & & & \\ & & \vdots & 1 & & & \\ \cdots & \cdots & C & & \ddots & & \\ & & & & & 1 & \\ 0 & & & & & & 1 \end{bmatrix} \end{array}$$

- Multiplying $[P(i, j : C)]$ to the matrix$[A]$ from its left side may yield addition of C times the column j of column i matrix$[A]$ to the column i matrix$[A]$.
- *A*ll upper triangular matrix transformation process adopts this process.

Triangular Factorization 2

- Accordingly, having a coefficient matrix $[A]$, an upper triangular matrix gained from $[A]$ as $[S]$, and an arbitrary elementary matrix $[Pn]$, the following relation can be established.

$$[P_n]\ldots[P_2][P_1][A] = [S]$$

- In expressing the previous matrix in the above expression may be,

$$\underbrace{\begin{bmatrix} 1 & & & 0 \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & -\frac{4}{3} & 1 \end{bmatrix}}_{step3} \underbrace{\begin{bmatrix} 1 & & & 0 \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & \frac{5}{14} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ 0 & 1 & & \\ 0 & -\frac{8}{7} & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{step2} \underbrace{\begin{bmatrix} 1 & & & 0 \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ 0 & 1 & & \\ \frac{1}{5} & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & 0 \\ -\frac{4}{5} & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{step1}$$

$$\begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & 0 & \frac{15}{7} & -\frac{20}{7} \\ 0 & 0 & 0 & \frac{5}{6} \end{bmatrix}$$

Triangular Factorization 3

- If we evaluate the product of elementary matrix at each step, we can find the value of diagonal term to be 1, 0 in all upper triangular section, and values are not 0 in corresponding rows for the lower triangular section.
- In addition, we may obtain the same matrix regardless of the multiplication order.

step 1.

$$\begin{bmatrix} 1 & & & \text{\Large 0} \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \text{\Large 0} \\ 0 & 1 & & \\ \frac{1}{5} & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \text{\Large 0} \\ -\frac{4}{5} & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \text{\Large 0} \\ -\frac{4}{5} & 1 & & \\ \frac{1}{5} & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

step 2.

$$\begin{bmatrix} 1 & & & \text{\Large 0} \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & \frac{5}{14} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \text{\Large 0} \\ 0 & 1 & & \\ 0 & -\frac{8}{7} & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \text{\Large 0} \\ 0 & 1 & & \\ 0 & -\frac{8}{7} & 1 & \\ 0 & \frac{5}{14} & 0 & 1 \end{bmatrix}$$

step 3.

$$\begin{bmatrix} 1 & & & \text{\huge 0} \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & -\frac{4}{3} & 1 \end{bmatrix}$$

Elementary Matrix Product and Inverse Matrix

- In general, the product of the matrix having fixed j in elementary matrix $[P(i, j : C)]$ with $i = j + 1 \sim n$, includes the diagonal term 1, row I of the column i + 1 includes $Ci+1, Ci+2, ...Cn$ below them. The left over element is found 0.

$$[P(i+1, i : C_{i+1})][P(i+2, i : C_{i+2})] \dots [P(u, i : C_n)] =$$

$$\begin{array}{cc}
 & i\,\text{列} \\
i\,\text{行} & \begin{bmatrix}
1 & & \vdots & & & & 0 \\
 & \ddots & \vdots & & & & \\
\cdots & \cdots & 1 & & & & \\
 & & C_{i+1} & 1 & & & \\
 & & C_{i+2} & & \ddots & & \\
0 & & \vdots & & & \ddots & \\
 & & C_n & 0 & & & 1
\end{bmatrix}
\end{array}$$

- $[P(i, j : C)]$ has its inverse matrix $[P(i, j : -C)]$ thus, the inverse matrix can be obtained by changing the signs of $C_{i+1}, C_{i+2}, ...Cn$

$$\begin{bmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & C_{i+1} & 1 & & \\ 0 & & \vdots & & \ddots & \\ & & C_n & 0 & & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & -C_{i+1} & 1 & & \\ 0 & & \vdots & & \ddots & \\ & & -C_n & 0 & & 1 \end{bmatrix}$$

Triangular Factorization By Elementary Matrix 1

- For a convenience in later formulation, define $[Li]$ based on the following. This is the inverse matrix adopted in each steps.

$$[L_i] = \begin{array}{cc} & \begin{array}{cccccc} & & i\,列 & & & \end{array} \\ i\,行 & \begin{bmatrix} 1 & & \vdots & & 0 & \\ & \ddots & \vdots & & & \\ \cdots & \cdots & 1 & & & \\ & & L_{i+1,i} & 1 & & \\ 0 & & \vdots & & \ddots & \\ & & L_{n,i} & 0 & & 1 \end{bmatrix} \end{array}$$

Provided that,

$$L_{i+j,i} = \frac{A_{i+j,i}^{(i)}}{A_{i,i}^{(i)}}$$

- If we express the inverse matrix of $[Li]$ as $[Li^{-1}]$, then $[Li^{-1}]$ may be found in the following.

$$[L_i^{-1}] = \begin{array}{cc} & \begin{array}{cccccc} & & i & & & \end{array} \\ i & \begin{bmatrix} 1 & & \vdots & & 0 & \\ & \ddots & \vdots & & & \\ \cdots & \cdots & 1 & & & \\ & & -L_{i+1,i} & 1 & & \\ 0 & & \vdots & & \ddots & \\ & & -L_{n,i} & 0 & & 1 \end{bmatrix} \end{array}$$

Triangular Factorization By Elementary Matrix 2

- When we suppose the dimensions of coefficient matrix$[A]$ to be *n*, we can take this $[Li^{-1}]$ to express the forward elimination operation shows in the following, and $[A^{(n)}]$ is found in the upper triangular matrix.

$$[A^{(2)}] = [L_1^{-1}][A^{(1)}]$$
$$\vdots$$
$$[A^{(n)}] = [L_{n-1}^{-1}][A^{(n-1)}]$$

- Suppose $[S]$ = $[A^{(n)}]$, we may write as the following.

$$[L_{n-1}^{-1}]\ldots[L_2^{-1}][L_1^{-1}][A] = [S]$$

Triangular Factorization By Elementary Matrix 3

$$[A^{(1)}] = \begin{bmatrix} 5 & -4 & 1 & 0 \\ -4 & 6 & -4 & 1 \\ 1 & -4 & 6 & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix}$$

$$[L_1^{-1}] = \begin{bmatrix} 1 & & & \text{\Large 0} \\ -L_{2,1} & 1 & & \\ -L_{3,1} & 0 & 1 & \\ -L_{4,1} & 0 & 0 & 1 \end{bmatrix} \quad 但し \begin{cases} L_{2,1} = \dfrac{A_{2,1}^{(1)}}{A_{1,1}^{(1)}} = -\dfrac{4}{5} \\ L_{3,1} = \dfrac{A_{3,1}^{(1)}}{A_{1,1}^{(1)}} = \dfrac{1}{5} \\ L_{4,1} = \dfrac{A_{4,1}^{(1)}}{A_{1,1}^{(1)}} = \dfrac{0}{5} = 0 \end{cases}$$

$$^{-1}][A^{(1)}] = \begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & -\frac{16}{5} & \frac{29}{5} & -4 \\ 0 & 1 & -4 & 5 \end{bmatrix} = [A^{(2)}]$$

$$[L_2^{-1}] = \begin{bmatrix} 1 & & & \text{\Large 0} \\ 0 & 1 & & \\ 0 & -L_{3,2} & 1 & \\ 0 & -L_{4,2} & 0 & 1 \end{bmatrix} \quad 但し \begin{cases} L_{3,2} = \dfrac{A_{3,2}^{(2)}}{A_{2,2}^{(2)}} = -\dfrac{8}{7} \\ L_{4,2} = \dfrac{A_{4,2}^{(2)}}{A_{2,2}^{(2)}} = \dfrac{5}{14} \end{cases}$$

$$[L_2^{-1}][A^{(2)}] = \begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & 0 & \frac{15}{7} & -\frac{20}{7} \\ 0 & 0 & -\frac{20}{7} & \frac{65}{14} \end{bmatrix} = [A^{(3)}]$$

$$[L_3^{-1}] = \begin{bmatrix} 1 & & & \text{\Large 0} \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & -L_{4,3} & 1 \end{bmatrix} \quad 但し\ L_{4,3} = \frac{A_{4,3}^{(3)}}{A_{3,3}^{(3)}} = -\frac{4}{3}$$

$$[L_3^{-1}][A^{(3)}] = \begin{bmatrix} 5 & -4 & 1 & 0 \\ 0 & \frac{14}{5} & -\frac{16}{5} & 1 \\ 0 & 0 & \frac{15}{7} & -\frac{20}{7} \\ 0 & 0 & 0 & \frac{5}{6} \end{bmatrix} = [A^{(4)}] = [S]$$

Triangular Factorization By Elementary Matrix 4

- Forward elimination,

$$[L_{n-1}^{-1}]\dots[L_2^{-1}][L_1^{-1}][A]=[S]$$

- By multiplying $[L_{n-1}]$, $[L_{n-2}]$, . . . , $[L_2]$, $[L_1]$ from the right side of the equation above in order, we can have a following equation.

$$[A]=[L_1][L_2]\dots[L_{n-1}][S]$$

- Here, $[L]$ is defined as,

$$[L]=[L_1][L_2]\dots[L_{n-1}]$$

- When $[L]$ is written out with its components,

$$[L]=\begin{bmatrix} 1 & & & & & & \\ & & & & & 0 & \\ L_{2,1} & 1 & & & & & \\ L_{3,1} & L_{3,2} & \ddots & & & & \\ & & & 1 & & & \\ \vdots & \vdots & & L_{i+1,i} & \ddots & & \\ & & & \vdots & & 1 & \\ L_{n,1} & L_{n,2} & \dots & L_{n,i} & \dots & L_{n,n-1} & 1 \end{bmatrix}$$

- $[A]$ can be expressed in the following with adopting $[L]$,

$$[A] = [L][S]$$

We can observe that the matrix is being decomposed into the product form of lower triangular matrix$[L]$
and upper triangular matrix$[S]$. This is called a triangular factorization.

Triangular Factorization By Elementary Matrix 5

- A diagonal term in $[S]$ is used as a denominator of division in evaluating unknown vector and on the way in triangular factorization. A list of diagonal terms $[S]$ is defined as diagonal matrix $[D]$, then it is often decomposed further as $[S] = [D][U]$.
- $[D]$, $[U]$ are written in its components,

$$
\begin{aligned}
[S] &= \begin{bmatrix} S_{11} & S_{12} & \dots & S_{1n} \\ & S_{22} & & \vdots \\ & & \ddots & \vdots \\ 0 & & & S_{nn} \end{bmatrix} \\
&= \begin{bmatrix} S_{11} & & & 0 \\ & S_{22} & & \\ & & \ddots & \\ 0 & & & S_{nn} \end{bmatrix} \begin{bmatrix} S_{11}/S_{11} & S_{12}/S_{11} & \dots & S_{1n}/S_{11} \\ & S_{22}/S_{22} & \dots & S_{2n}/S_{22} \\ & & \ddots & \vdots \\ 0 & & & S_{nn}/S_{nn} \end{bmatrix} \\
&= \begin{bmatrix} S_{11} & & & 0 \\ & S_{22} & & \\ & & \ddots & \\ 0 & & & S_{nn} \end{bmatrix} \begin{bmatrix} 1 & S_{12}/S_{11} & \dots & S_{1n}/S_{11} \\ & 1 & & S_{2n}/S_{22} \\ & & \ddots & \vdots \\ 0 & & & 1 \end{bmatrix}
\end{aligned}
$$

- Therefore,

$$[D] = \begin{bmatrix} S_{11} & & & \text{\huge 0} \\ & S_{22} & & \\ & & \ddots & \\ \text{\huge 0} & & & S_{nn} \end{bmatrix}$$

$$[U] = \begin{bmatrix} 1 & S_{12}/S_{11} & \dots & S_{1n}/S_{11} \\ & 1 & & S_{2n}/S_{22} \\ & & \ddots & \vdots \\ \text{\huge 0} & & & 1 \end{bmatrix}$$

Triangular Factorization By Elementary Matrix 6

- In this operation, $[A]$ is eventually factorized as following equation.

$$[A] = [L][D][U]$$

- By an implementation of such formulation, we can construct the loop index in symmetry in the process of coding.
- If $[A]$ is a symmetric matrix, then $[U] = [L^T]$. Therefore, $[A]$ is factorized in the following equation.

$$[A] = [L][D][L^T] = [U^T][D][U]$$

Solving Triangular Matrix and System of Linear Equation 1

- By factorization into $[A] = [L][D][U]$, the system of linear equation $[A]\{b\} = \{c\}$ adopts inverse matrix to both sides in turn to obtain the solution.

$$
\begin{aligned}
[L][D][U]\{b\} &= \{c\} \\
[D][U]\{b\} &= [L^{-1}]\{c\} \\
[U]\{b\} &= [D^{-1}][L^{-1}]\{c\} \\
\{b\} &= [U^{-1}][D^{-1}][L^{-1}]\{c\}
\end{aligned}
$$

- First, evaluate $[L^{-1}]\{c\}$. Put it as $\{x\} = [L^{-1}]\{c\}$. $\{x\}$ represents the solution of system of linear equations having a lower triangular matrix as the coefficient matrix.

$$
[L]\{x\} = \{c\}
$$

- In such system of linear equations, the solutions may be obtained by forward substitution, in other words, by a process exactly inverse the backward substitution we studied earlier. Specifically,

$$
\begin{aligned}
x_1 &= c_1 \\
x_2 &= c_2 - L_{21}x_1 \\
x_3 &= c_3 - L_{31}x_1 - L_{32}x_2 \\
&\vdots
\end{aligned}
$$

- Obviously, it is possible to consider the forward elimination in the same way we did for the backward elimination, yet it is not adoptable in the skyline method, and therefore leave out the explanation.

Solving Triangular Matrix and System of Linear Equation 2

- In the next step, we evaluate $[D^{-1}][L^{-1}]\{c\}$. Using $\{x\}$ obtained before, we can get, $[D^{-1}][L^{-1}]\{c\}$ = $[D^{-1}]\{x\}$, and which can be placed by $\{y\} = [D^{-1}]\{x\}$. Now we find $\{y\}$ to be the solution for the system of linear equations, which having the diagonal matrix as coefficient matrix.

$$[D]\{y\} = \{x\}$$

- In such system of linear equations, the solutions can be obtained in a simple division. Specifically,

$$y_i = x_i / D_{ii} \ (i = 1 \sim n)$$

- Finally, evaluate $[U^{-1}][D^{-1}][L^{-1}]\{c\}$. Using $\{y\}$ obtained earlier, we get $\{b\} = [U^{-1}]\{y\}$, then we find $\{b\}$ to be the solutions of the system of linear equations, which having the upper triangular matrix as coefficient matrix.

$$[U]\{b\} = \{y\}$$

- The solutions for such system of linear equations can be obtained by either the backward elimination or the backward substitution.

Solving Triangular Matrix and System of Linear Equation 3

- Having completed the triangular factorization, this process can be implemented in far less amount of calculation compared to the calculations involved in triangular factorization. Moreover, in the case where there are many different right side vectors present, it is possible to obtain the solution efficiently by using the identical coefficient matrix.
- In the example examined on the topic of the forward elimination, coefficient matrix $[A^{(4)}]$ corresponds to $[S]$,while the right side vector $\{c^{(4)}\}$ corresponds to $\{x\} = [L^{-1}]\{c\}$.
- In actual calculations for the example above may be given as,

$$[L] = \begin{bmatrix} 1 & & & 0 \\ -\frac{4}{5} & 1 & & \\ \frac{1}{5} & -\frac{8}{7} & 1 & \\ 0 & \frac{5}{14} & -\frac{4}{3} & 1 \end{bmatrix}$$

$$[D] = \begin{bmatrix} 5 & & & 0 \\ & \frac{14}{5} & & \\ & & \frac{15}{7} & \\ 0 & & & \frac{5}{6} \end{bmatrix}$$

$$[U] = [L^T] = \begin{bmatrix} 1 & -\frac{4}{5} & \frac{1}{5} & 0 \\ & 1 & -\frac{8}{7} & \frac{5}{14} \\ & & 1 & -\frac{4}{3} \\ 0 & & & 1 \end{bmatrix}$$

$$[L^{-1}] = [L_{n-1}^{-1}] \cdots [L_2^{-1}][L_1^{-1}] = \begin{bmatrix} 1 & & & \\ \frac{4}{5} & 1 & & 0 \\ \frac{5}{7} & \frac{8}{7} & 1 & \\ \frac{2}{3} & \frac{7}{6} & \frac{4}{3} & 1 \end{bmatrix}$$

$$[L^{-1}]\{c\} = \begin{bmatrix} 1 & & & 0 \\ \frac{4}{5} & 1 & & \\ \frac{5}{7} & \frac{8}{7} & 1 & \\ \frac{2}{3} & \frac{7}{6} & \frac{4}{3} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{8}{7} \\ \frac{7}{6} \end{bmatrix} = \{c^{(4)}\}$$

Triangular Matrix Product 1

- Gauss elimination includes various code styles, yet what we consider in discussion here is just one of them.
  1. Easy to grasp the calculation.
  2. Basically, constructed in the same technique with skyline method, which introduced in later.
- So far, we indicated that the general matrix can be implemented triangular factorization, but here we start from already factorized matrix. Consider now the following 4 × 4 matrix.

$$[L] = \begin{bmatrix} 1 & & & 0 \\ L_{21} & 1 & & \\ L_{31} & L_{32} & 1 & \\ L_{41} & L_{42} & L_{43} & 1 \end{bmatrix} [D] = \begin{bmatrix} D_{11} & & & 0 \\ & D_{22} & & \\ & & D_{33} & \\ 0 & & & D_{44} \end{bmatrix} [U] = \begin{bmatrix} 1 & U_{12} & U_{13} & U_{14} \\ & 1 & U_{23} & U_{24} \\ & & 1 & U_{34} \\ 0 & & & 1 \end{bmatrix}$$

- By taking the product of [L], [D], [U], then given by [A] = [L][D][U] ,

$$[A] =$$
$$\begin{bmatrix} D_{11} & D_{11}U_{12} & D_{11}U_{13} & D_{11}U_{14} \\ L_{21}D_{11} & L_{21}D_{11}U_{12} + D_{22} & L_{21}D_{11}U_{13} + D_{22}U_{23} & L_{21}D_{11}U_{14} + D_{22}U_{24} \\ L_{31}D_{11} & L_{31}D_{11}U_{12} + L_{32}D_{22} & L_{31}D_{11}U_{13} + L_{32}D_{22}U_{23} + D_{33} & L_{31}D_{11}U_{14} + L_{32}D_{22}U_{24} + D_{33}U_{34} \\ L_{41}D_{11} & L_{41}D_{11}U_{12} + L_{42}D_{22} & L_{41}D_{11}U_{13} + L_{42}D_{22}U_{23} + L_{43}D_{33} & L_{41}D_{11}U_{14} + L_{42}D_{22}U_{24} + L_{43}D_{33}U_{34} + D_{44} \end{bmatrix}$$

Triangular Matrix Product 2

- This is generally expressed by using index ,

$$A_{11} = D_{11}$$

$$A_{1i} = D_{11} U_{1i} \qquad i = 2 \sim n$$

$$A_{i1} = L_{i1} D_{11} \qquad i = 2 \sim n$$

Where i ≥ 2

$$A_{ij} = D_{ii} U_{ij} + \sum_{k=1}^{i-1} L_{ik} D_{kk} U_{kj} \quad (i < j : \text{上三角側})$$

$$A_{ij} = L_{ij} D_{jj} + \sum_{k=1}^{j-1} L_{ik} D_{kk} U_{kj} \quad (i > j : \text{下三角側})$$

$$A_{ii} = D_{ii} + \sum_{k=1}^{i-1} L_{ik} D_{kk} U_{ki} \qquad (i = j)$$

Triangular Factorization Procedure

Based on this relationship,[L], [D], [U] are calculated in turn,

$$j = 1$$

$$D_{11} = A_{11}$$

$$j = 2$$

$$\begin{cases} U_{12} = A_{12}/D_{11} \\ L_{21} = A_{21}/D_{11} \end{cases}$$

$$D_{22} = A_{22} - L_{21}D_{11}U_{12}$$

$j = 3$

$$\begin{cases} U_{13} = A_{13}/D_{11} \\ L_{31} = A_{31}/D_{11} \end{cases}$$

$$\begin{cases} U_{23} = (A_{23} - L_{21}D_{11}U_{13})/D_{22} \\ L_{32} = (A_{32} - L_{31}D_{11}U_{12})/D_{22} \end{cases}$$

$$D_{33} = A_{33} - L_{31}D_{11}U_{13} - L_{32}D_{22}U_{23}$$

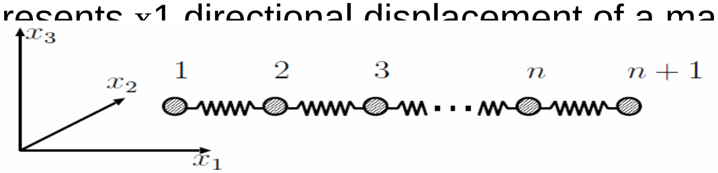
$$\vdots$$

Boundary Condition Handling

- As figure below indicates, consider a system where $n + 1$ mass points, which parallel to x_1 axis, are connected with n springs. (mass at each mass point: $m_i (i = 1 \sim n + 1)$, spring constant for each spring:
$k_i (i = 1 \sim n)$)
- Suppose each mass point is restricted to behave in xi direction only, with external force $f_i (i = 1 \sim n + 1)$ acted on each mass point, the equations of motion for each mass point can be given by,

$$
\begin{aligned}
m_1 \ddot{u}_1 &= f_1 & & + k_1(u_2 - u_1) \\
m_2 \ddot{u}_2 &= f_2 & + k_1(u_1 - u_2) & + k_2(u_3 - u_2) \\
m_3 \ddot{u}_3 &= f_3 & + k_2(u_2 - u_3) & + k_3(u_4 - u_3) \\
&\vdots & & \\
m_n \ddot{u}_n &= f_n & + k_{n-1}(u_{n-1} - u_n) & + k_n(u_{n+1} - u_n) \\
m_{n+1} \ddot{u}_{n+1} &= f_{n+1} & + k_n(u_n - u_{n+1}) &
\end{aligned}
$$

Where $u_i (i = 1 \sim n + 1)$ represents x1 directional displacement of a mass point m_i.



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Boundary Condition Handling 2

- In the matrix representation,

$$\begin{bmatrix} k_1 & -k_1 & & & & \\ -k_1 & k_1+k_2 & -k_2 & & & \\ & -k_2 & k_2+k_3 & -k_3 & & \\ & & & \ddots & & \\ & & & -k_{n-1} & k_{n-1}+k_n & -k_n \\ & & & & -k_n & k_{n+1} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \\ u_{n+1} \end{Bmatrix} = \begin{Bmatrix} f_1 - m_1\ddot{u}_1 \\ f_2 - m_2\ddot{u}_2 \\ f_3 - m_3\ddot{u}_3 \\ \vdots \\ f_n - m_n\ddot{u}_n \\ f_{n+1} - m_{n+1}\ddot{u}_{n+1} \end{Bmatrix}$$

- Consider now a case where . $\ddot{u}_1 = \ddot{u}_2 = \cdots = \ddot{u}_n = \ddot{u}_{n+1} = 0, \dot{u}_1 = \dot{u}_2 = \cdots = \dot{u}_n = \dot{u}_{n+1} = 0$
  Although, physically, a balance of force should be kept, the point of balance in the whole system is not yet determined. The coefficient matrix include its rank as n.

Boundary Condition Handling 3

- Generally, the stiffness matrix used to deal with stationary problems in finite element analysis can be gained by discritization of the force balance equation, and the position of balance do not reflect on the equations.
- Stiffness matrix is unique thus the solution cannot be obtained unless adding other conditions such as $u_1 = 0$.
- There are cases where the displacements such as $u_j = a$ in some points are considered as arbitrary set points by acting some virtual external force to have problem setting.
- Then u_j is known and takes the value that is not 0, while f_j is unknown.
- In finite element analysis, to handle a part of some displacements , which is intrinsically unknown quantity, as known quantity is generally called boundary condition handling, especially with know quantity 0, is called enforced displacement.

Boundary Condition Handling 4

- Consider now for the following system of linear equations.

$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & & & \vdots \\ \vdots & & & \vdots \\ A_{n1} & \dots & \dots & A_{nn} \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix} = \begin{Bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{Bmatrix}$$

- Suppose we have known information on the right hand side except for Cj ,and bj. To have the matrix formulation in normal system of linear equations formulation, which is given by,

$$\begin{cases} A_{11} + A_{12}b_2 + \cdots + A_{1j}b_j + \cdots + A_{1n}b_n & = C_1 \\ A_{21} + A_{22}b_2 + \cdots + A_{2j}b_j + \cdots + A_{2n}b_n & = C_2 \\ \vdots & \vdots \\ A_{j1} + A_{j2}b_2 + \cdots + A_{jj}b_j + \cdots + A_{jn}b_n & = C_j \\ \vdots & \vdots \\ A_{n1} + A_{n2}b_2 + \cdots + A_{nj}b_j + \cdots + A_{nn}b_n & = C_n \end{cases}$$

- In above, the equation in the column j contains unknown quantity Cj in the right side, thus Cj is obtained after the ordinary unknown quantity $bi(i = 1 \sim j - 1, j+$

$1 \sim n$) are obtained based on the following.

$$C_j = A_{j1}b_1 + A_{j2}b_2 + \cdots + Ajjb_j + \cdots + A_{jn}b_n$$

It is distinct in nature compared to other equations in the column, thus $bi(i = 1 \sim j - 1, j + 1 \sim n)$ should be excluded from the process.

Boundary Condition Handling 5

- Consider the $n-1$ equations without column j .

$$
\left\{
\begin{array}{ccccccccccccc}
A_{11} & + & A_{12}b_2 & + & \cdots & + & A_{1j}b_j & + & \cdots & + & A_{1n}b_n & = & C_1 \\
A_{21} & + & A_{22}b_2 & + & \cdots & + & A_{2j}b_j & + & \cdots & + & A_{2n}b_n & = & C_2 \\
 & & & & & & \vdots & & & & & & \vdots \\
A_{j-1\,1} & + & A_{j-1\,2}b_2 & + & \cdots & + & A_{j-1\,j}b_j & + & \cdots & + & A_{j-1\,n}b_n & = & C_{j-1} \\
A_{j+1\,1} & + & A_{j+1\,2}b_2 & + & \cdots & + & A_{j+1\,j}b_j & + & \cdots & + & A_{j-1\,n}b_n & = & C_{j+1} \\
 & & & & & & \vdots & & & & & & \vdots \\
A_{n1} & + & A_{n2}b_2 & + & \cdots & + & A_{nj}b_j & + & \cdots & + & A_{nn}b_n & = & C_n
\end{array}
\right.
$$

- In the equation above, $A_{1j}bj$, $A2jbj$, ・・・, $Anjbj$ are known value and should be transpositioned into the right hand side.

$$
\left\{
\begin{array}{cccccc}
A_{11}+A_{12}b_2 & +\cdots+ & A_{1\,j-1}b_{j-1}+A_{1\,j+1}b_{j+1} & +\cdots+ & A_{1n}b_n & = & C_1-A_{1j}b_j \\
A_{21}+A_{22}b_2 & +\cdots+ & A_{2\,j-1}b_{j-1}+A_{2\,j+1}b_{j+1} & +\cdots+ & A_{2n}b_n & = & C_2-A_{2j}b_j \\
 & & \vdots & & & & \vdots \\
A_{j-1\,1}+A_{j-1\,2}b_2 & +\cdots+ & A_{j-1\,j-1}b_{j-1}+A_{j-1\,j+1}b_{j+1} & +\cdots+ & A_{j-1\,n}b_n & = & C_{j-1}-A_{j-1\,j}b_j \\
A_{j+1\,1}+A_{j+1\,2}b_2 & +\cdots+ & A_{j+1\,j-1}b_{j-1}+A_{j+1\,j+1}b_{j+1} & +\cdots+ & A_{j+1\,n}b_n & = & C_{j+1}-A_{j+1\,j}b_j \\
 & & \vdots & & & & \vdots \\
A_{n1}+A_{n2}b_2 & +\cdots+ & A_{n\,j-1}b_{j-1}+A_{n\,j+1}b_{j+1} & +\cdots+ & A_{nn}b_n & = & C_n-A_{nj}b_j
\end{array}
\right.
$$

Boundary Condition Handling 6

- To formally express in matrix, we have the following.
- Coefficient matrix consists of the matrix without column *j* and the row *j* .
- Unknown vector represents the one without column *j*.
- The right hand side is consisted of the initial left hand side subtracted by *bj* times the initial coefficient matrix row *j* without the column *j*.

$$
\begin{bmatrix}
A_{11} & \dots & A_{1\,j-1} & A_{1\,j+1} & \dots & A_{1n} \\
A_{21} & \dots & A_{2\,j-1} & A_{2\,j+1} & \dots & A_{2n} \\
\vdots & & & & & \\
A_{j-1\,1} & \dots & A_{j-1\,j-1} & A_{j+1\,j+1} & \dots & A_{j-1\,n} \\
A_{j+1\,1} & \dots & A_{j+1\,j+1} & A_{j+1\,j+1} & \dots & A_{j+1\,n} \\
\vdots & & & & & \\
A_{n1} & \dots & A_{n\,j-1} & A_{n\,j+1} & \dots & A_{nn}
\end{bmatrix}
\left\{ \begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_{j-1} \\ b_{j+1} \\ \vdots \\ b_n \end{matrix} \right\}
=
\left\{ \begin{matrix} C_1 \\ C_2 \\ \vdots \\ C_{j-1} \\ C_{j+1} \\ \vdots \\ C_n \end{matrix} \right\}
- b_j
\left\{ \begin{matrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{j-1\,j} \\ A_{j+1\,j} \\ \vdots \\ A_{nj} \end{matrix} \right\}
$$

- Where *bj* is 0, the right hand side the second term can be eliminated.
- The operation above stays the same with multiple known *bj*. Suppose *bj*(1), *bj*(2) ,・・・, *bj*(*m*) are know. From the system of linear equations, exclude the equations in column *j*(1), *j*(2), ・・・, *j*(*m*).Then transposition the terms, which include *bj*(1),*bj*(2),・・・,*bj*(*m*) in the left side to the right hand side. Based on the fact, the matrix can be obtained.

Gauss Elimination : Quiz

Conducting triangular factorization to the matrix $[A]$, according to gauss elimination, the following can be obtained. Show the process of triangular factorization.

$$[A] = \begin{bmatrix} 4 & -3 & 1 & 0 \\ -3 & 5 & -3 & 1 \\ 1 & -3 & 5 & -3 \\ 0 & 1 & -3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -3 & 1 & 0 \\ -3 & 5 & -3 & 1 \\ 1 & -3 & 5 & -3 \\ 0 & 1 & -3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{3}{4} & 1 & 0 & 0 \\ \frac{1}{4} & -\frac{9}{11} & 1 & 0 \\ 0 & \frac{4}{11} & -\frac{3}{4} & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & \frac{11}{4} & 0 & 0 \\ 0 & 0 & \frac{32}{11} & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -\frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{9}{11} & \frac{4}{11} \\ 0 & 0 & 1 & -\frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$