Nonlinear
Finite Element Method

13/12/2004

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- Lectures include discussion of the nonlinear finite element method.
- It is preferable to have completed “Introduction to Nonlinear Finite Element Analysis” available in summer session.
- If not, students are required to study on their own before participating this course. Reference:Toshiaki.,Kubo. “Introduction: Tensor Analysis For Nonlinear Finite Element Method” (Hisennkei Yugen Yoso no tameno Tensor Kaiseki no Kiso),Maruzen.
- Lecture references are available and downloadable at http://www.sml.k.u-tokyo.ac.jp/members/nabe/lecture2004 They should be posted on the website by the day before scheduled meeting, and each students are expected to come in with a copy of the reference.
- Lecture notes from previous year are available and downloadable, also at http://www.sml.k.u.tokyo.ac.jp/members/nabe/lecture2003 You may find the course title, ”Advanced Finite Element Method” but the contents covered are the same I will cover this year.
- I will assign the exercises from this year, and expect the students to hand them in during the following lecture. They are not the requirements and they will not be graded, however it is important to actually practice calculate in deeper understanding the finite element method.
- For any questions, contact me at nabe@sml.k.u-tokyo.ac.jp

Nonlinear Finite Element Method Lecture Schedule

1. 10/ 4 Finite element analysis in boundary value problems and the differential equations
2. 10/18 Finite element analysis in linear elastic body
3. 10/25 Isoparametric solid element (program)
4. 11/ 1 Numerical solution and boundary condition processing for system of linear equations（with exercises）
5. 11/ 8 Basic program structure of the linear finite element method(program)
6. 11/15 Finite element formulation in geometric nonlinear problems(program)
7. 11/22 Static analysis technique、hyperelastic body and elastic-plastic material for nonlinear equations (program)
8. 11/29 Exercises for Lecture7
9. 12/ 6 Dynamic analysis technique and eigenvalue analysis in the nonlinear equations
10. 12/13 Constitutive element
11. 12/20 Numerical solution— skyline method、iterative method for the system of linear equations
12. 1/17 ALE finite element fluid analysis
13. 1/24 ALE finite element fluid analysis

Dynamic Analysis Technique

- Equation of motion at time $t+\Delta t$,

$$\boldsymbol{M} \cdot {}^{t+\Delta t}\ddot{\boldsymbol{u}} + \boldsymbol{C} \cdot {}^{t+\Delta t}\dot{\boldsymbol{u}} + {}^{t+\Delta t}\boldsymbol{Q} = {}^{t+\Delta t}\boldsymbol{F} \tag{1}$$

- $\boldsymbol{M}$ represents a mass matrix and $\boldsymbol{C}$ represents a damping matrix.
- Here, we state the direct time integration method, which integrate the equation as it is by direction of time.
- In linear problems, it is sometimes analyzed by superposition of natural mode, and it is called the mode superposition method.
- Given a solution at time t , a method of evaluating an accurate solution of $t+\Delta t$ is called an implicit method.（linear acceleration method and Newmark-β method, etc.), while a method of making a prediction based on a solution of t is called an explicit method.（central difference）
- Normally, $\boldsymbol{M}$ and $\boldsymbol{C}$ stay constant regardless of time, yet $\boldsymbol{Q}$ changes.
- Linearize the inner vectors then adopt the Newton-Raphson method.

$$\boldsymbol{M} \cdot {}^{t+\Delta t}\ddot{\boldsymbol{u}}^{(k)} + \boldsymbol{C} \cdot {}^{t+\Delta t}\dot{\boldsymbol{u}}^{(k)} + {}^{t+\Delta t}\boldsymbol{K}^{(k-1)}\Delta\boldsymbol{u}^{(k)} = {}^{t+\Delta t}\boldsymbol{F} - {}^{t+\Delta t}\boldsymbol{Q}^{(k-1)} \tag{2}$$

$${}^{t+\Delta t}\boldsymbol{u}^{(k)} = {}^{t}\boldsymbol{u}^{(k-1)} + \Delta\boldsymbol{u}^{(k)} \tag{3}$$

$${}^{t+\Delta t}\dot{\boldsymbol{u}}^{(k)} = {}^{t}\dot{\boldsymbol{u}}^{(k-1)} + \Delta\dot{\boldsymbol{u}}^{(k)} \tag{4}$$

$${}^{t+\Delta t}\ddot{\boldsymbol{u}}^{(k)} = {}^{t}\ddot{\boldsymbol{u}}^{(k-1)} + \Delta\ddot{\boldsymbol{u}}^{(k)} \tag{5}$$

In this way, the above includes the unknown quantities $\Delta\boldsymbol{u}^{(k)}, \Delta\dot{\boldsymbol{u}}^{(k)}, \Delta\ddot{\boldsymbol{u}}^{(k)}$ thus, usually the three quantities are related under some supposition, and put them all together to make single variable in analyzation.

Linear Acceleration Method

- Linear acceleration method is hardly used in actual analysis.
- Literally, it supposes the linear change in acceleration in time t and $t+\Delta t$.

$$ {}^{t+\tau}\ddot{\boldsymbol{u}} = {}^{t}\ddot{\boldsymbol{u}} + \frac{\tau}{\Delta t}({}^{t+\tau}\ddot{\boldsymbol{u}} - {}^{t}\ddot{\boldsymbol{u}}) \tag{6} $$

- Having τ as a variable, and integrate the equation above then substitute $\tau = \Delta t$ may yield,

$$ {}^{t+\Delta t}\dot{\boldsymbol{u}} = {}^{t}\dot{\boldsymbol{u}} + \frac{\Delta t}{2}\left({}^{t+\Delta t}\ddot{\boldsymbol{u}} + {}^{t}\ddot{\boldsymbol{u}}\right) \tag{7} $$

$$ {}^{t+\Delta t}\boldsymbol{u} = {}^{t}\boldsymbol{u} + \Delta t\, {}^{t}\dot{\boldsymbol{u}} + \frac{\Delta t^2}{3}{}^{t}\ddot{\boldsymbol{u}} + \frac{\Delta t^2}{6}{}^{t+\Delta t}\ddot{\boldsymbol{u}} \tag{8} $$

- Here we have,

$$ \Delta\dot{\boldsymbol{u}}^{(k)} = \frac{\Delta t}{2}\Delta\ddot{\boldsymbol{u}}^{(k)} \tag{9} $$

$$ \Delta\boldsymbol{u}^{(k)} = \frac{\Delta t^2}{6}\Delta\ddot{\boldsymbol{u}}^{(k)} \tag{10} $$

Newmark - β Method 1

- Newmark-β method is first introduced by Newmark in 1959 and since then it has contributed various benefits in applications to the area of dynamics.
- This technique is signified as unconditional stability of time integration scheme, thus the dynamic analysis can be conducted with a great time increment step Δt .
- Moreover, introduction of numerical damping not only to the higher order modes but also to the lower order modes is possible.
- The equation of motion at time $t+\Delta t$ is given by,

$$\boldsymbol{M} \cdot {}^{t+\Delta t}\ddot{\boldsymbol{u}} + \boldsymbol{C} \cdot {}^{t+\Delta t}\dot{\boldsymbol{u}} + {}^{t+\Delta t}\boldsymbol{Q} = {}^{t+\Delta t}\boldsymbol{F} \tag{11}$$

- Newmark-β method assumes the relation of deformation and velocity at time $t+\Delta t$ from t as in the following equation.

$$ {}^{t+\Delta t}\dot{\boldsymbol{u}} = {}^{t}\dot{\boldsymbol{u}} + \Delta t \left[\gamma \, {}^{t+\Delta t}\ddot{\boldsymbol{u}} + (1-\gamma) \, {}^{t}\ddot{\boldsymbol{u}}\right] \tag{12}$$

$$ {}^{t+\Delta t}\boldsymbol{u} = {}^{t}\boldsymbol{u} + \Delta t \, {}^{t}\dot{\boldsymbol{u}} + \Delta t^2 \left\{ \left(\frac{1}{2} - \beta\right) {}^{t}\ddot{\boldsymbol{u}} + \beta \, {}^{t+\Delta t}\ddot{\boldsymbol{u}} \right\} \tag{13}$$

Newmark - β Method 2

$$ {}^{t+\Delta t}\dot{\boldsymbol{u}} = {}^{t}\dot{\boldsymbol{u}} + \Delta t \left[\gamma \, {}^{t+\Delta t}\ddot{\boldsymbol{u}} + (1-\gamma) \, {}^{t}\ddot{\boldsymbol{u}} \right] \tag{14} $$

$$ {}^{t+\Delta t}\boldsymbol{u} = {}^{t}\boldsymbol{u} + \Delta t \, {}^{t}\dot{\boldsymbol{u}} + \Delta t^2 \left\{ \left(\frac{1}{2} - \beta \right) {}^{t}\ddot{\boldsymbol{u}} + \beta \, {}^{t+\Delta t}\ddot{\boldsymbol{u}} \right\} \tag{15} $$

- If we set $\gamma = \frac{1}{2}$, $\beta = \frac{1}{6}$ then the equation may coincide with the equation in linear acceleration method.
- If we set , $\gamma = \frac{1}{2}$, $\beta = \frac{1}{4}$ in above equation, then ${}^{t+\Delta t}\dot{\boldsymbol{u}}$ and ${}^{t+\Delta t}\boldsymbol{u}$ are given by the following equations.

$$ {}^{t+\Delta t}\dot{\boldsymbol{u}} = {}^{t}\dot{\boldsymbol{u}} + \frac{\Delta t}{2} ({}^{t+\Delta t}\ddot{\boldsymbol{u}} - {}^{t+\Delta t}\dot{\boldsymbol{u}} \tag{16} $$

$$ {}^{t+\Delta t}\boldsymbol{u} = {}^{t}\boldsymbol{u} + \Delta t \, {}^{t}\dot{\boldsymbol{u}} + \frac{\Delta t^2}{4} ({}^{t}\ddot{\boldsymbol{u}} + {}^{t+\Delta t}\ddot{\boldsymbol{u}}) \tag{17} $$

- Apparently, we assume this relation to take a constant average velocity at *t* to $t+\Delta t$.

$$ {}^{t+\tau}\ddot{\boldsymbol{u}} = \frac{1}{2} ({}^{t}\ddot{\boldsymbol{u}} + {}^{t+\Delta t}\ddot{\boldsymbol{u}}) \qquad (0 \leq \tau \leq \Delta t) \tag{18} $$

The combinations for $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$ are idiomatically called the trapezoidal rule.

Newmark - β Method 3

- In transforming the relational expression of Newmark-β method with a deformation at $t+\Delta t$ in addition to the deformation, acceleration and velocity at t, we can express the acceleration and the velocity at $t+\Delta t$ can be given by,

$$^{t+\Delta t}\ddot{\boldsymbol{u}} = \frac{1}{\beta\Delta t^2}(^{t+\Delta t}\boldsymbol{u} - {}^{t}\boldsymbol{u}) - \frac{1}{\beta\Delta t}{}^{t}\dot{\boldsymbol{u}} - \left(\frac{1}{2\beta} - 1\right){}^{t}\ddot{\boldsymbol{u}} \tag{19}$$

$$\begin{aligned}
^{t+\Delta t}\dot{\boldsymbol{u}} &= {}^{t}\dot{\boldsymbol{u}} + \left\{(1-\gamma)^{t}\ddot{\boldsymbol{u}} + \gamma\left[\frac{1}{\beta\Delta t^2}(^{t+\Delta t}\boldsymbol{u} - {}^{t}\boldsymbol{u}) - \frac{1}{\beta\Delta t}{}^{t}\dot{\boldsymbol{u}} - (\frac{1}{2\beta} - 1)^{t}\ddot{\boldsymbol{u}}\right]\right\}\Delta t \\
&= {}^{t}\dot{\boldsymbol{u}} + \Delta t(1-\gamma)^{t}\ddot{\boldsymbol{u}} + \frac{\gamma}{\beta\Delta t}(^{t+\Delta t}\boldsymbol{u} - {}^{t}\boldsymbol{u}) - \frac{\gamma}{\beta}{}^{t}\dot{\boldsymbol{u}} - \gamma\left(\frac{1}{2\beta} - 1\right)\Delta t^{t}\ddot{\boldsymbol{u}} \\
&= \frac{\gamma}{\beta\Delta t}(^{t+\Delta t}\boldsymbol{u} - {}^{t}\boldsymbol{u}) + \left(1 - \frac{\gamma}{2\beta}\right){}^{t}\dot{\boldsymbol{u}} + \Delta t\left(1 - \frac{\gamma}{2\beta}\right){}^{t}\ddot{\boldsymbol{u}}
\end{aligned} \tag{20}$$

- In linear problems we can use the linear stiffness matrix $\boldsymbol{K}$ and express the inner force $\boldsymbol{Q}^{t+\Delta t}$,

$$^{t+\Delta t}\boldsymbol{Q} = \boldsymbol{K}{}^{t+\Delta t}\boldsymbol{u} \tag{21}$$

- Executing equation of Newmark-β method in linear system,

$$\begin{aligned}
&\left(\frac{1}{\beta\Delta t^2}\boldsymbol{M} + \frac{\gamma}{\beta\Delta t}\boldsymbol{C} + \boldsymbol{K}\right){}^{t+\Delta t}\boldsymbol{u} \\
&\qquad = {}^{t+\Delta t}\boldsymbol{F} + \boldsymbol{M}\left[\frac{1}{\beta\Delta t^2}{}^{t}\boldsymbol{u} + \frac{1}{\beta\Delta t}{}^{t}\dot{\boldsymbol{u}} + \left(\frac{1}{2\beta} - 1\right){}^{t}\ddot{\boldsymbol{u}}\right] \\
&\qquad\qquad + \boldsymbol{C}\left[\frac{\gamma}{\beta\Delta t}{}^{t}\boldsymbol{u} + \left(\frac{\gamma}{\beta} - 1\right){}^{t}\dot{\boldsymbol{u}} + \left(\frac{\gamma}{2\beta} - 1\right)\Delta t^{t}\ddot{\boldsymbol{u}}\right]
\end{aligned} \tag{22}$$

Newmark - β Method 4

- Executing equation of Newmark- β method in linear system,

$$\left(\frac{1}{\beta\Delta t^2}\boldsymbol{M}+\frac{\gamma}{\beta\Delta t}\boldsymbol{C}+\boldsymbol{K}\right){}^{t+\Delta t}\boldsymbol{u}$$
$$={}^{t+\Delta t}\boldsymbol{F}+\boldsymbol{M}\left[\frac{1}{\beta\Delta t^2}{}^{t}\boldsymbol{u}+\frac{1}{\beta\Delta t}{}^{t}\dot{\boldsymbol{u}}+\left(\frac{1}{2\beta}-1\right){}^{t}\ddot{\boldsymbol{u}}\right]$$
$$+\boldsymbol{C}\left[\frac{\gamma}{\beta\Delta t}{}^{t}\boldsymbol{u}+\left(\frac{\gamma}{\beta}-1\right){}^{t}\dot{\boldsymbol{u}}+\left(\frac{\gamma}{2\beta}-1\right)\Delta t\,{}^{t}\ddot{\boldsymbol{u}}\right] \quad (23)$$

- Linear analysis programming can be implemented if $\left(\frac{1}{\beta\Delta t^2}\boldsymbol{M}+\frac{\gamma}{\beta\Delta t}\boldsymbol{C}+\boldsymbol{K}\right)$ is adopted in stead of the usual stiffness matrix.

- Under linear system, given $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$, the solution is stable regardless of the value in time step Δt.

- While , $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$ are given, if the value Δt takes below the certain value, we may obtain a guarantee in solution stability, in other words, which implies a conditional stability.

- However, where $\gamma > \frac{1}{2}$, the numerical damping effect may take place.

Newmark - β Method 5

- In the same way we handled stationary problems, iteration $(k = 2, 3, \cdots)$ on Newton-Raphson method based on the following equation can be implemented.

$$\boldsymbol{M}^{t+\Delta t}\ddot{\boldsymbol{u}}^{(k)} + \boldsymbol{C}^{t+\Delta t}\dot{\boldsymbol{u}}^{(k)} + {}^{t+\Delta t}\boldsymbol{K}^{(k-1)}\Delta\boldsymbol{u}^{(k)} = {}^{t+\Delta t}\boldsymbol{F} - {}^{t+\Delta t}\boldsymbol{Q}^{(k-1)} \quad (24)$$

$${}^{t+\Delta t}\boldsymbol{u}^{(k)} = {}^{t+\Delta t}\boldsymbol{u}^{(k-1)} + \Delta\boldsymbol{u}^{(k)} \quad (25)$$

$${}^{t+\Delta t}\dot{\boldsymbol{u}}^{(k)} = {}^{t+\Delta t}\dot{\boldsymbol{u}}^{(k-1)} + \Delta\dot{\boldsymbol{u}}^{(k)} \quad (26)$$

$${}^{t+\Delta t}\ddot{\boldsymbol{u}}^{(k)} = {}^{t+\Delta t}\ddot{\boldsymbol{u}}^{(k-1)} + \Delta\ddot{\boldsymbol{u}}^{(k)} \quad (27)$$

Where signs in upper right side (k) indicate the k *th* iteration.

- In conducting iteration by Newmark–β method, if deformation is given unknown, take the difference of acceleration, deformation and velocity relational expression at the k *th* and $k-1\mathrm{th}$ iteration.

$$\Delta\dot{\boldsymbol{u}}^{(k)} = \frac{\gamma}{\beta\Delta t}\Delta\boldsymbol{u}^{(k)} \quad (28)$$

$$\Delta\ddot{\boldsymbol{u}}^{(k)} = \frac{1}{\beta\Delta t^2}\Delta\boldsymbol{u}^{(k)} \quad (29)$$

System of linear equation with unknown variable $\Delta \boldsymbol{u}^{(k)}$

$$\left(\frac{1}{\beta\Delta t^2}\boldsymbol{M} + \frac{\gamma}{\beta\Delta t}\boldsymbol{C} + \boldsymbol{K}\right)\Delta\boldsymbol{u}^{(k)}$$

$$= {}^{t+\Delta t}\boldsymbol{F} - {}^{t+\Delta t}\boldsymbol{Q}^{(k-1)} - \boldsymbol{M}\,{}^{t+\Delta t}\ddot{\boldsymbol{u}}^{(k-1)} - \boldsymbol{C}\,{}^{t+\Delta t}\dot{\boldsymbol{u}}^{(k-1)} \quad (30)$$

Newmark - β Method 6

- Newmark-β method implementation procedures are indicated in the following.
  1. Initial value calculation
     (a) Obtain the mass matrix $\boldsymbol{M}$ and the damping matrix $\boldsymbol{C}$.
     (b) Calculate the following invariables.

$$– \quad a_0 = \frac{1}{\beta \Delta t^2} \qquad a_1 = \frac{\gamma}{\beta \Delta t} \qquad a_2 = \frac{1}{\beta \Delta t}$$

$$– \quad a_3 = \frac{1}{2\beta} - 1 \qquad a_4 = \frac{\gamma}{\beta} - 1 \qquad a_5 = \frac{\gamma}{2\beta} - 1$$

$$– \quad a_6 = \gamma \Delta t \qquad a_7 = (1-\gamma)\Delta t$$

  2. Calculations at each step
     (a)Calculate the significant stiffness matrix at time $t+\Delta t$.

$$– \quad \boldsymbol{K}_{\text{eff}} = a_0 \boldsymbol{M} + a_1 \boldsymbol{C} + \boldsymbol{K}$$

     (b) Calculate the significant load at time $t+\Delta t$.

$$– \quad {}^{t+\Delta t}\boldsymbol{F}_{\text{eff}} = {}^{t+\Delta t}\boldsymbol{F} - {}^{t}\boldsymbol{Q} + \boldsymbol{M}(a_2{}^{t}\dot{\boldsymbol{u}} + a_3{}^{t}\ddot{\boldsymbol{u}}) + \boldsymbol{C}(a_4{}^{t}\dot{\boldsymbol{u}} + a5\ddot{\boldsymbol{u}})$$

     (c) Evaluate the deformation of time $t+\Delta t$: $t+\Delta$

$$– \quad \boldsymbol{K}_{\text{eff}} \Delta \boldsymbol{u} = {}^{t}\boldsymbol{F}_{\text{eff}} \qquad {}^{t+\Delta t}\boldsymbol{u} = {}^{t}\boldsymbol{u} + \Delta \boldsymbol{u}$$

     (d) For nonlinear problems, iteration can be calculated as,

       i. $\Delta U^{(0)} = \Delta U$, $i = 0$ と置く.

       ii. $i = i + 1$

iii. Calculate the significant load for the $i-1$ th iteration.

- $^{t+\Delta t}\boldsymbol{F}_{\text{eff}}^{(i-1)} = {}^{t+\Delta t}\boldsymbol{F} - \boldsymbol{M}^{t+\Delta t}\ddot{\boldsymbol{u}}^{(i-1)} - \boldsymbol{C}^{t+\Delta t}\dot{\boldsymbol{u}}^{(i-1)} - {}^{t+\Delta t}\boldsymbol{Q}^{(i-1)}$

iv.Calculate the deformation increment for the i th iteration.

- $\boldsymbol{K}_{\text{eff}}\Delta\boldsymbol{u}^{(i)} = {}^{t+\Delta t}\boldsymbol{F}_{\text{eff}}^{(i-1)}$

v. Calculate the deformation increment of single increment step.

- $\Delta\boldsymbol{u} = \Delta\boldsymbol{u} + \Delta\boldsymbol{u}^{(i)}$

vi. If convergence exists, then the deformation, velocity and acceleration at time $t+\Delta t$ can be evaluated in the following calculation.

- $^{t+\Delta t}\ddot{\boldsymbol{u}} = a_3\Delta\boldsymbol{u} + a_4{}^{t}\dot{\boldsymbol{u}} + a_5{}^{t}\ddot{\boldsymbol{u}}$
- $^{t+\Delta t}\dot{\boldsymbol{u}} = {}^{t}\dot{\boldsymbol{u}} + a_6{}^{t}\ddot{\boldsymbol{u}} + a_7{}^{t+\Delta t}\ddot{\boldsymbol{u}}$
- $^{t+\Delta t}\boldsymbol{u} = {}^{t}\boldsymbol{u} + \Delta\boldsymbol{u}$