1

Cognition and Activity in Geometry

Non-Euclidean Geometry and Time-Space

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Mathematical Sciences

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Geometry

① Science of figures on a plane or a space

↓

- Euclid   axiomatic method
- Descartes   analytic geometry

② Questioning the "real nature" of structures of a plane or a space themselves

↓

- non-Euclidean geometry
- Gauss, Riemann

③ Concept of "a plain space" released from structures

↓

- manifold
- phase space, topology

④ Dynamic image of space

an example of

④ Dynamic Space Image

- Cut, paste and change "spaces themselves" and create a new space

- When "an initial value" is set to the space structure, we can observe the space moving on its own. (the solution to Poincare conjecture by Perelman)

- Elasticized to ∞-dimension and broken down to "sums" and "products", the space shows algebraic structure in its framework.
(algebraic topology)

- Sometimes, a structure that plain space naturally possesses shows its profound nature.

Today...

- Sphere geometry and non-Euclidean geometry
- Minkowski space
- Manifold and phase space
- (Special theory of relativity)

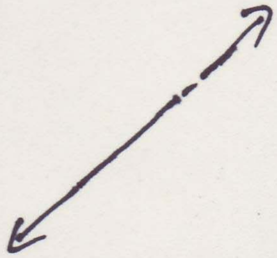
" ① Study of figures on a plane or a space " 1

--- Euclidean eometry [proof]

(1)

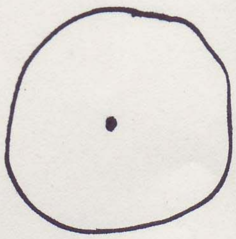
There is only one line that connects two different points.

(2)



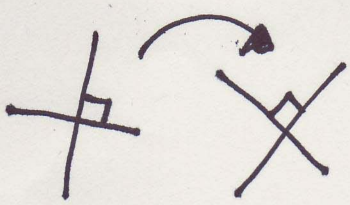
The line can be extended to any length (infinitely).

(3)



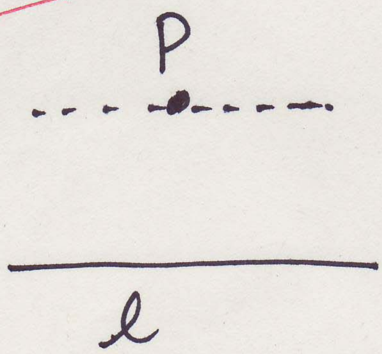
If a center and a radius are given, a circle can be drawn.

(4)



Right angles are congruent.

(5)



[proposition concerning parallel lines]

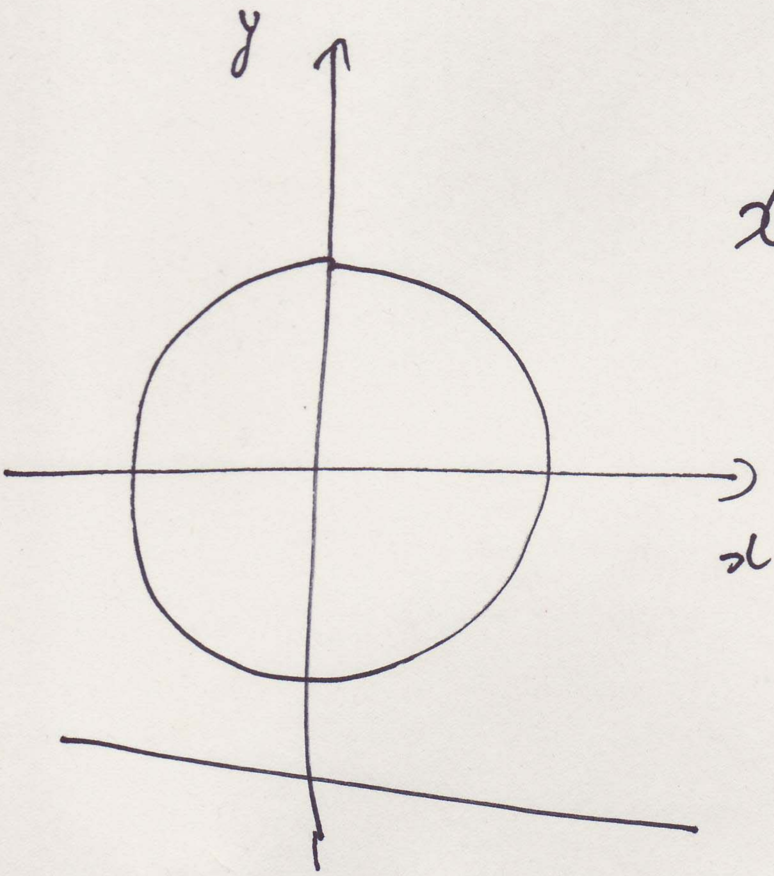
⇕

There is only one line that is "parallel" to ℓ and runs through P.

① Study of figures on a plane or a space

" 2

--- analytic geometry [algebraic calculation]



$x^2+y^2=1$ ··· circle

$ax+by+c=0$

··· line

- A point on a plane is expressed by using a coordinate (x, y).

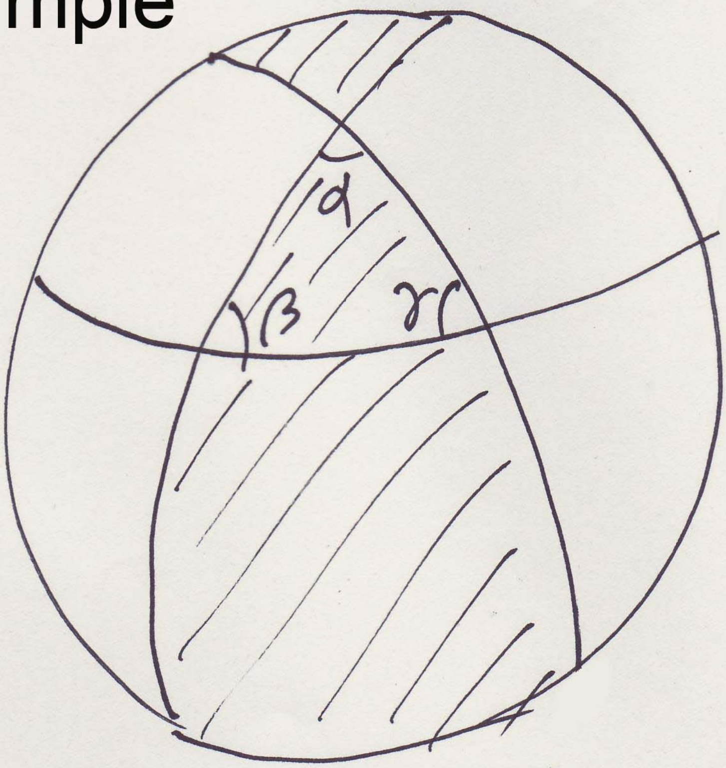
- In a space, (x, y, z).

Spherical Geometry

--- can be developed by using Euclidean geometry or analytical geometry.

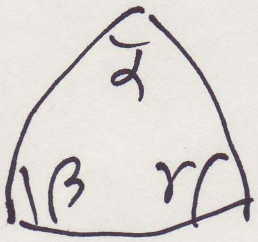
Example

Sphere with a radius r



What is the area of a spheric triangle?

$$\frac{2\alpha}{2\pi} \times 4\pi r^2 = 4\alpha r^2$$



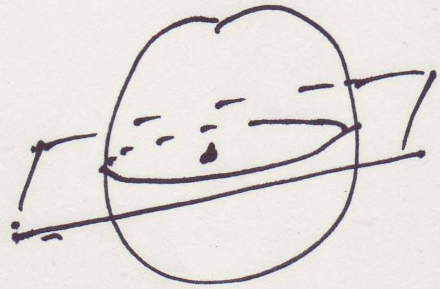
$$= \frac{4\alpha r^2 + 4\beta r^2 + 4\gamma r^2 - 4\pi r^2}{4}$$

$$= \boxed{(\alpha + \beta + \gamma - \pi) \cdot r^2}$$

Q. Is spherical geometry "non-Euclidean geometry"?

What if "line" "length" and "angle" are considered as below?

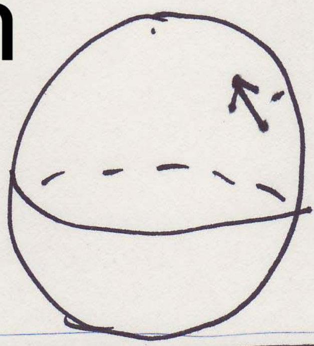
- **line** $:=$ **big circle**



this means

$\boxed{\text{a plane that passes origin} \cap \text{a sphere}}$

- (minim) **length**

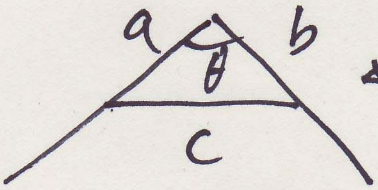


$$\cdots \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

normal length

$$\boxed{\text{length} \fallingdotseq \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}}$$

- **angle**



← minim triangle

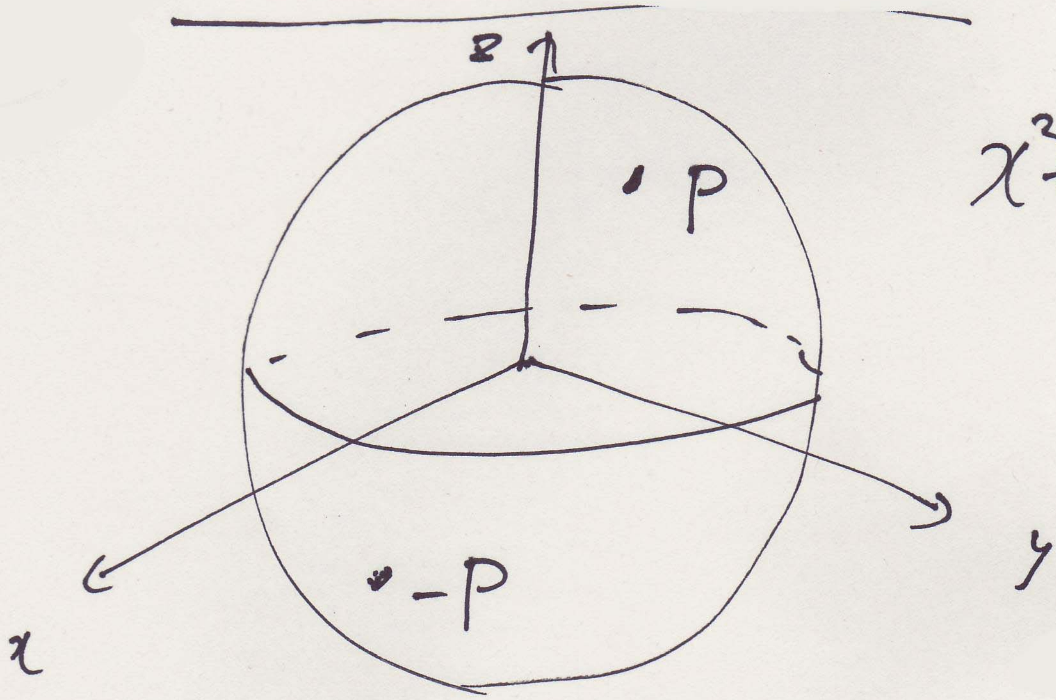
normal angle

$$\boxed{\cos\theta = \frac{a^2 + b^2 - c^2}{2ab}}$$

Problem 1

Two different lines should be crossed at only one point.

However, they cross at two points!



$$x^2+y^2+z^2=r^2$$

Solution The pair $\{P, -P\}$ should be defined again as "a single point"!

$\Longrightarrow$ $\boxed{\text{projective plane}} := \{\{P, -P\}\}_{P \in \text{sphere}}$

Problem Two

"A finite line" is wanted to be elongated infinitely.

However, it goes round and comes back.

Spherical geometry was well-known in the late 18th century (at the latest).

(map drawing method, sphere astronomy etc.)

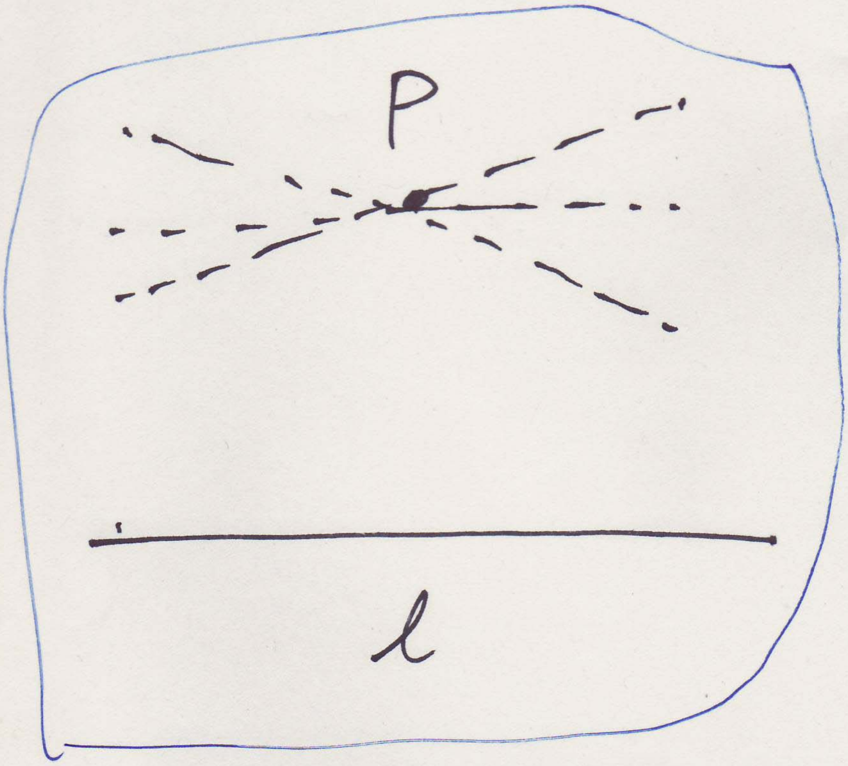
However, it was never considered to be "non-Euclidean geometry".

This must be because of

problem two above.

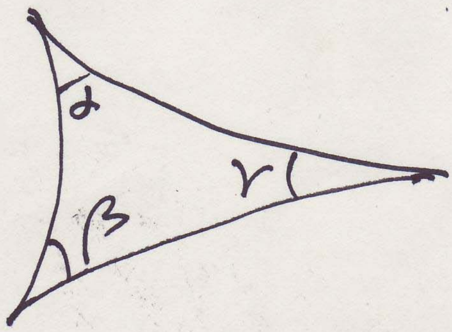
non-Euclidean geometry

19th century (Lobachevsky, Bolyai
Gauss)



Suppose that there are more than two (infinite) lines that do not intersect with ℓ, and pass P.

In that world, sine theorem and cosine theorem were proved, too, and development of an original geometry was recognized.



Ex. $\propto \pi-(\alpha+\beta+\gamma)$

Gauss's Insight

- Does this world

  follow Euclidean geometry?
  or...
  follow non-Euclidean geometry?

- This cannot be solved by thinking.

- An experiment has to be conducted to judge which is correct.

Gauss actually attempted to do this by surveying a huge triangle connecting three mountains.

Looking back again on spherical geometry...

(1) If we forget "the possibility of infinite elongation of a line", it can be considered as just a parallel system to non-Euclidean geometry discovered by Gauss and others.

(2) Sphere geometry has its spheres right in front of our eyes. (the Earth, the celestial globe).

However, there were no such things as "right in front of our eyes" in the non-Euclidean geometry that Gauss discovered.

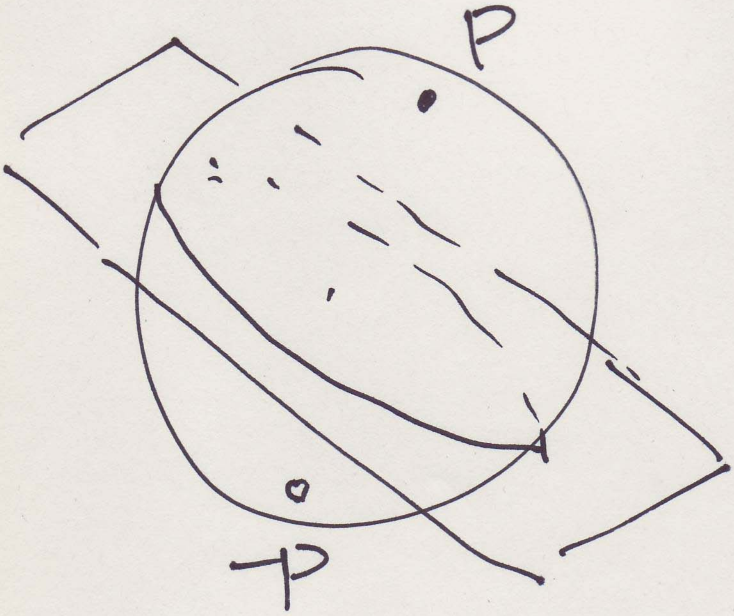
— the gift of sophisticated abstract thinking

--- Later, the "model" for non-Euclidean geometry was found.

(Klein, Poincare ---)

To explain the "model",

let us review spherical geometry.



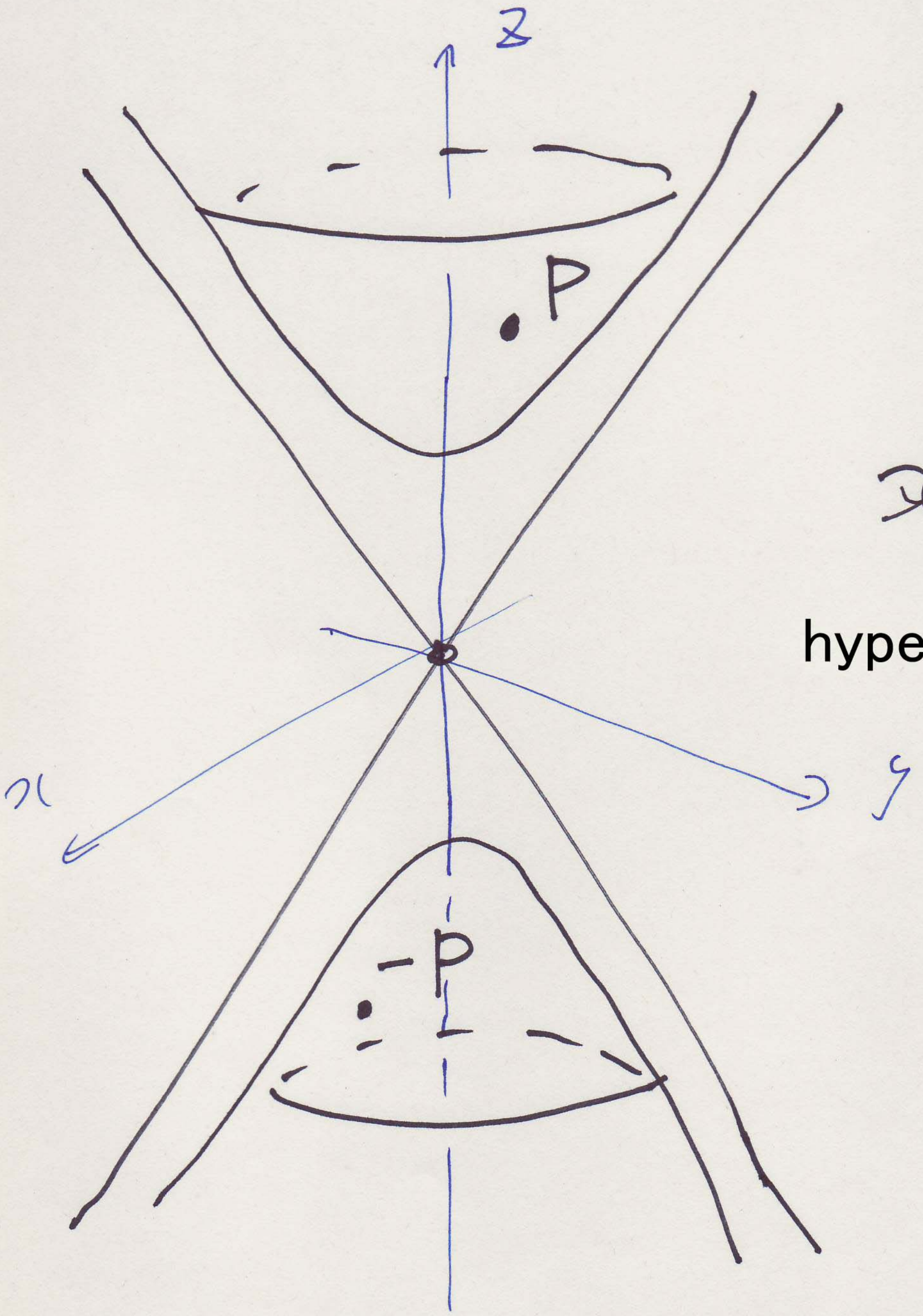
$$x^2+y^2+z^2=1$$

point = pair $\{P, -P\}$

line = plane that pass origin $\cap$ sphere

minim distance = $\sqrt{(\Delta x)^2+(\Delta y)^2+(\Delta z)^2}$

An example of non-Euclidean geometry models



$$x^2+y^2-z^2=-1$$

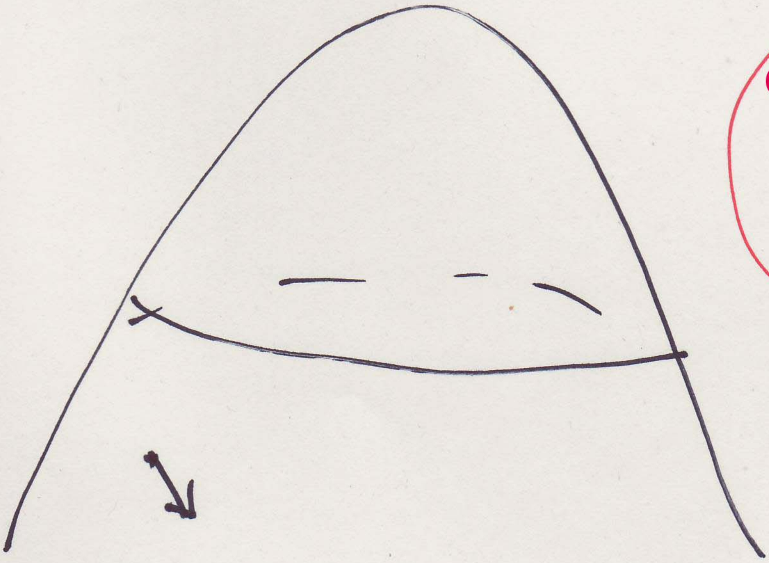
hyperboloid of two sheets

"point" = the pair $\{P, -P\}$

"line" = a plane that passes the origin $\cap$ hyperboloid

How to determine a distance

$$\begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$



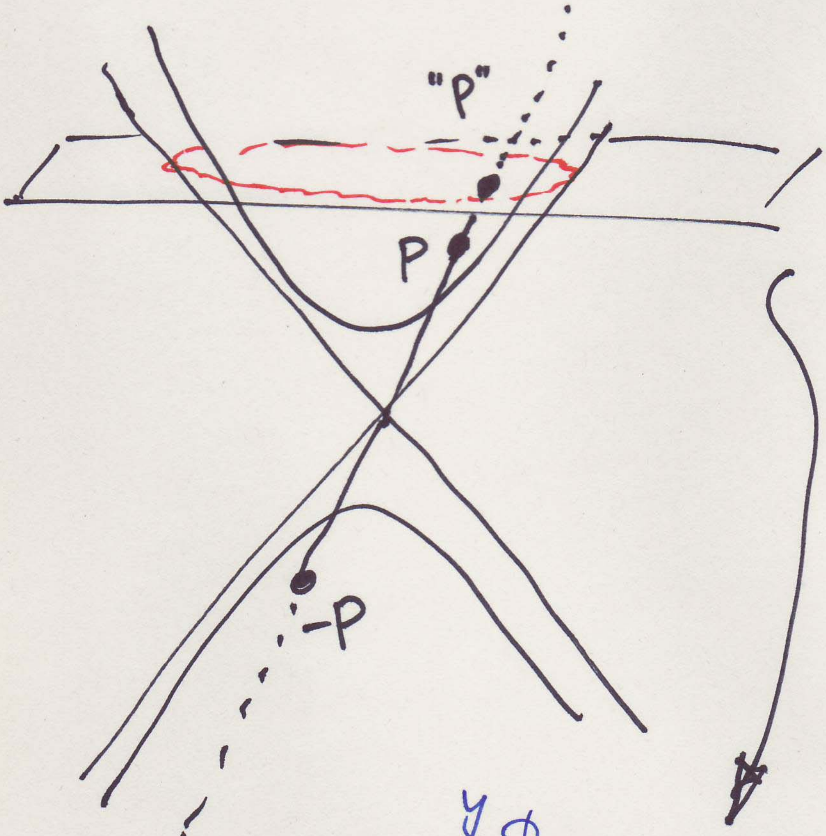
different from $\mathbb{R}^3$'s normal distance

$$\text{minimal distance} = \sqrt{\underbrace{(\Delta x)^2 + (\Delta y)^2 - (\Delta z)^2}}$$

If $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\begin{pmatrix} x+\Delta x \\ y+\Delta y \\ z+\Delta z \end{pmatrix}$ are on hyperboloid, inside $\sqrt{}$ is positive.

When projected to the plane Z=1

by a line that passes the origin ...



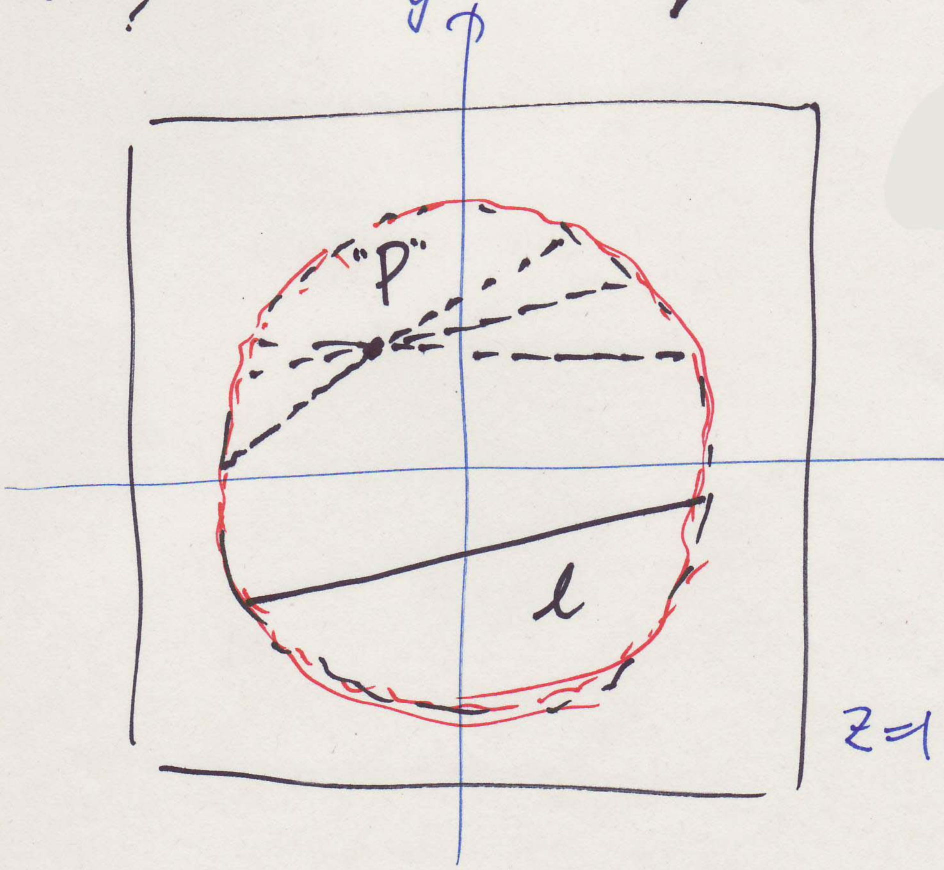
"line"

$\parallel$

a plane that passes the origin $\cap$ hyperboloid

$\downarrow$ projection

line $\cap \{x^2+y^2<1\}$



In case of spherical geometry

Every two points on a sphere are "coordinative",
and they exchange their locations by some rotations.

An example of spherical rotations

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 1 & & \\ & \cos\theta & -\sin\theta \\ & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(1) The equation $x^2+y^2+z^2=1$ is maintained.

(2) "Line" is transferred to a "line".

(3) minimal distance^2 $= (\Delta x)^2+(\Delta y)^2+(\Delta z)^2$ is maintained.

$\because$ $\sin^2\theta + \cos^2\theta = 1$

In non-Euclidean geometry "models",

Any two points on hyperboloids are "coordinative", and

by some "rotation" they exchange their positions.

an example of "rotation"

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} 1 & & \\ & \cosh\tau & \cosh\tau \\ & \sinh\tau & \sinh\tau \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(1) The equation $x^2+y^2-z^2=-1$ is maintained.

(2) "Line" is transferred to "line".

(3) $\text{minimal distance}^2 = (\Delta x)^2+(\Delta y)^2-(\Delta z)^2$ is maintained.

$\because)\quad -\sinh^2\tau + \cosh^2\tau = 1$

$$\left(\sinh\tau = \frac{e^{\tau}-e^{-\tau}}{2},\ \cosh\tau = \frac{e^{\tau}+e^{-\tau}}{2} \right)$$

Consistency of non-Euclidean geometry

- Before Lobachevsky, Bolyai, and Gauss, challenges were made to consider a logical conclusion on the assumption that parallel postulate is not true. (In the late 18th century, by Sacchari, Lambert etc.)

- However, the aim of these challenges was to derive contradiction.

- Lobachevsky, Bolyai, and Gauss had not proved that "contradiction cannot be derived", but there was a conclusive insight that a new geometry was being established.

- By the existence of analytic geometry model, consistency of non-Euclidean geometry boils down to that of Euclidean geometry.

Later ....

Let us look at two (independent) cases below :

A $\mathbb{R}^3$ is forgotten, and

the focus is on hyperboloid or sphere

B To $\mathbb{R}^3$,

$$(\Delta x)^2 + (\Delta y)^2 - (\Delta z)^2$$

is introduced and focused on.

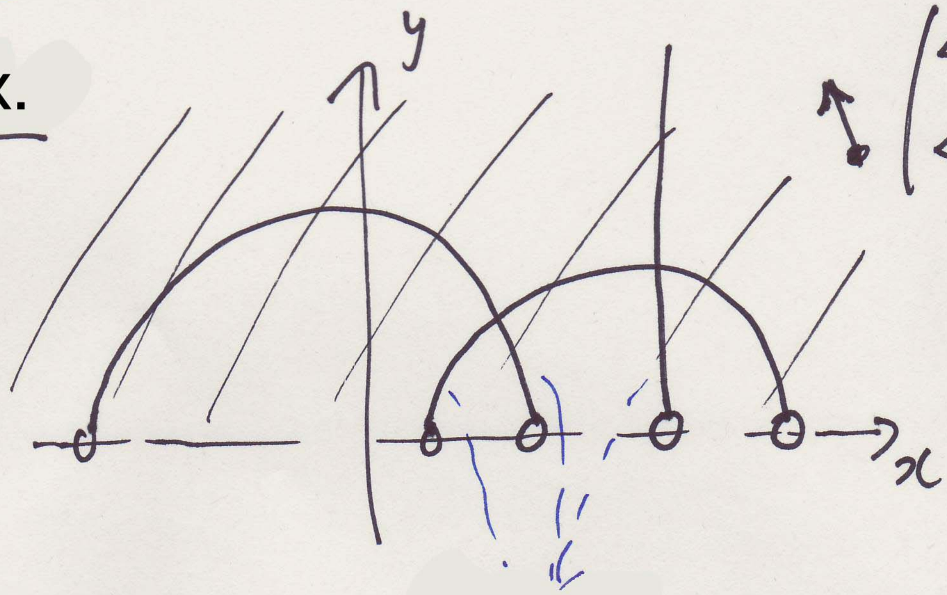
Let's look at further descriptions of each.

P Let us forget surroundings, and focus on "spaces" (sphere, hyperboloid) themselves.

(Gauss, Riemann ---)

There are various kinds of non-Euclidean models.

Ex.



$$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} \quad \text{distance} = \frac{\sqrt{(\Delta x)^2+(\Delta y)^2}}{y}$$

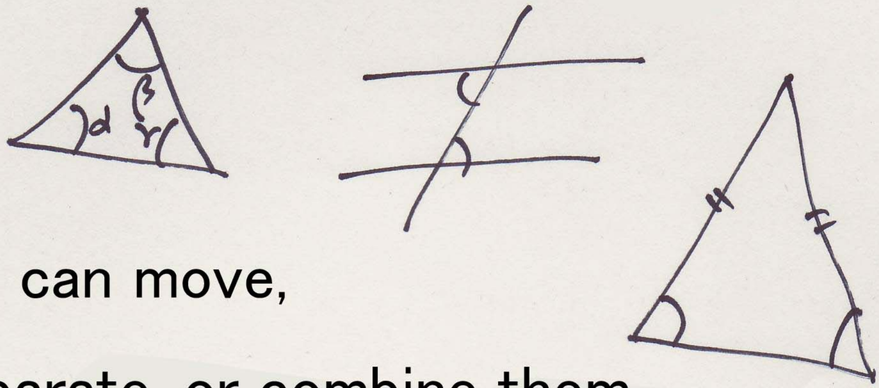
A model is just a model.

If there is an intrinsic geometry, thoughts can be developed more freely.

$\longrightarrow$

The classic image of geometry

Various geometrical subjects/phenomenon are in (non)-Euclidean spaces.



We can move, separate, or combine them.

[(Non)-Euclidean spaces are their background.]

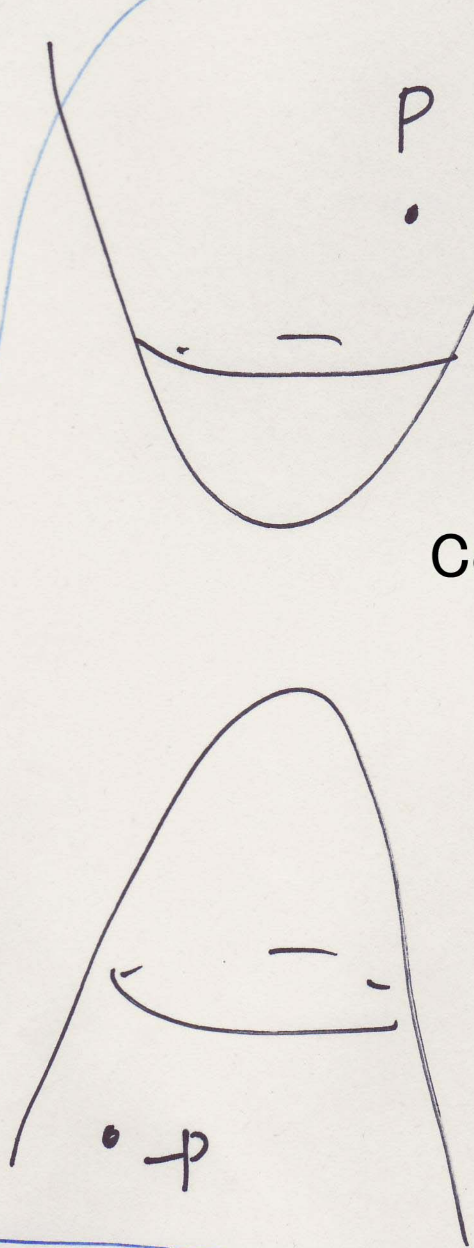
development

The new image of geometry

As a background to support geometrical subjects/phenomenon, (non)-Euclidean spaces themselves have various possibilities.

Spaces themselves can be moved, separated, or combined.

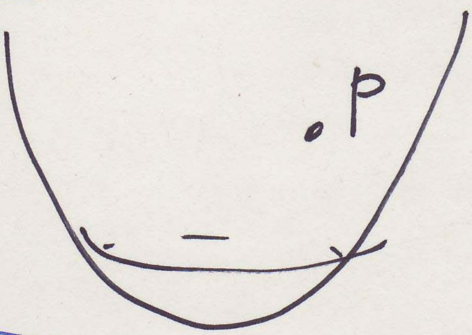
Let us reconsider "space itself".



Consider $\{p, -p\}$

as a "point".

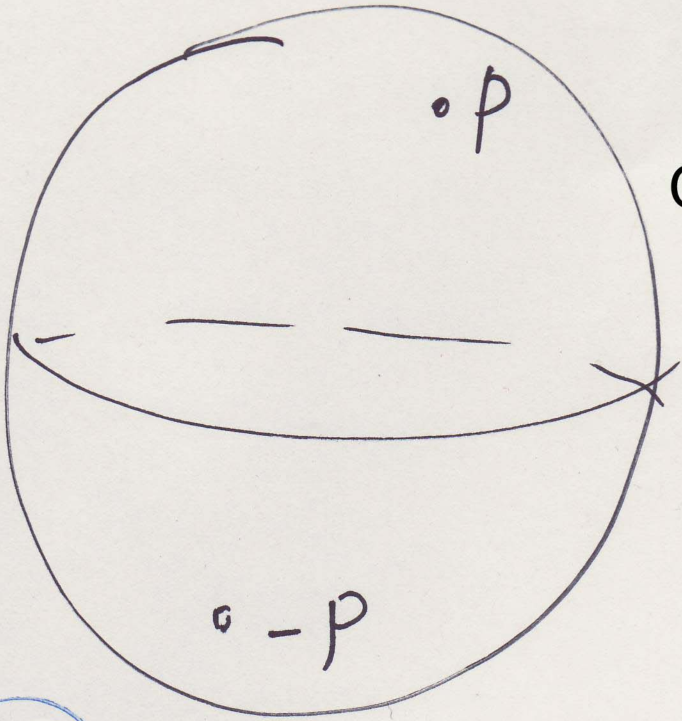
equal



p is a point.

(obediently)

(real) Projective plane



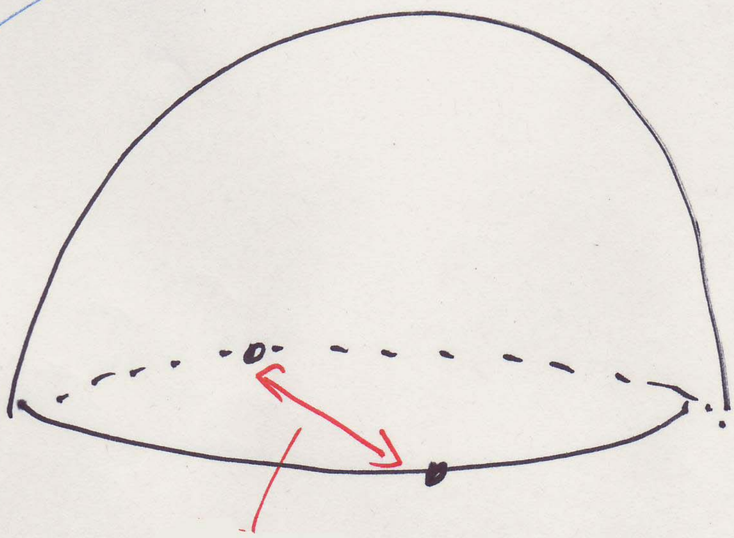
Consider $\{p, -p\}$

as a single

"point".

strict

equal



On the border of the

hemisphere,

identify two

opposite points

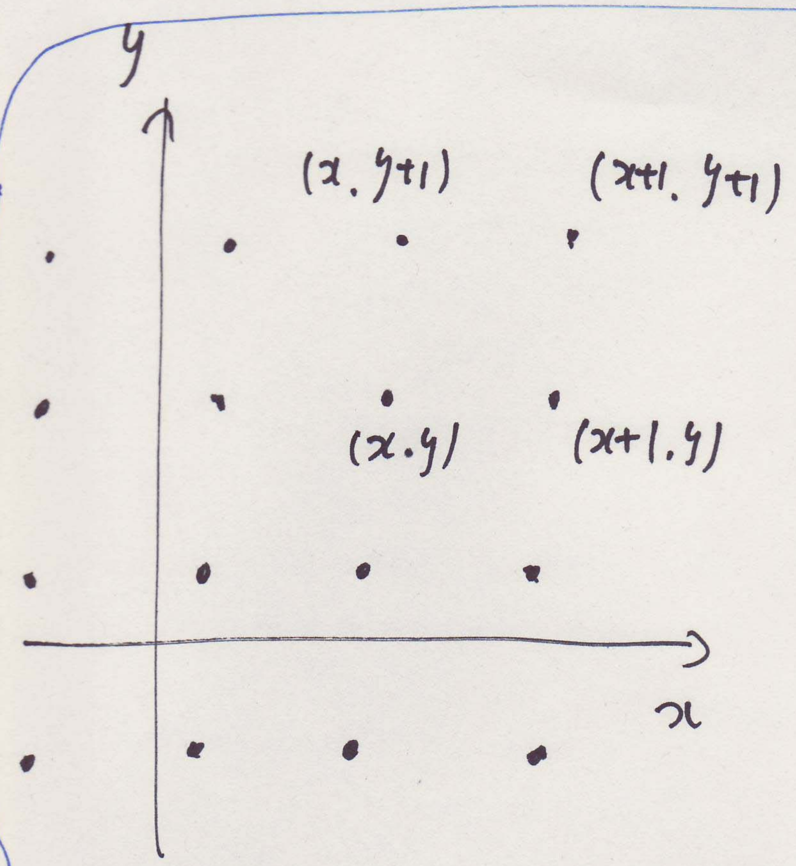
and

fold them together.

intuitive

can't be strict

A similar example (1) — Torus



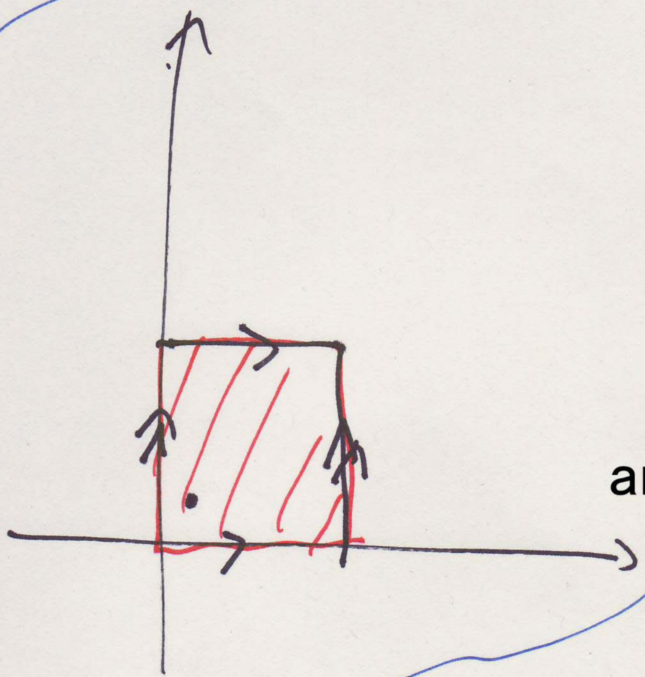
For each (x, y),

consider the set

$$\boxed{\{(x+n,\ y+m) \mid m, n \text{ integer}\}}$$

as a single point.

equal



Identify each opposite sides of a square and fold them together.

intuitive

(cannot be strict)

A similar example (2) Grassmann variety

In $\mathbb{R}^4$

the set of all possibilities of

all "two-dimensional sub-vector spaces"

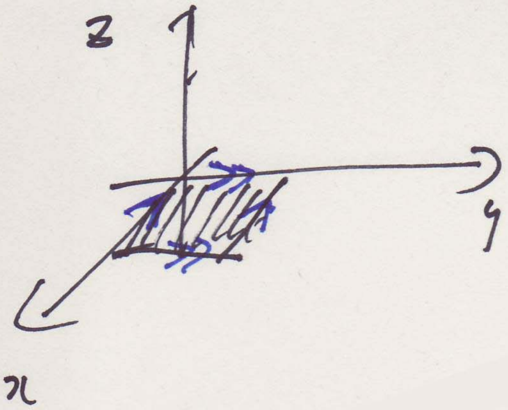
is called $Gr_2(\mathbb{R}^4)$.

In $Gr_2(\mathbb{R}^4)$, a "point" is a single

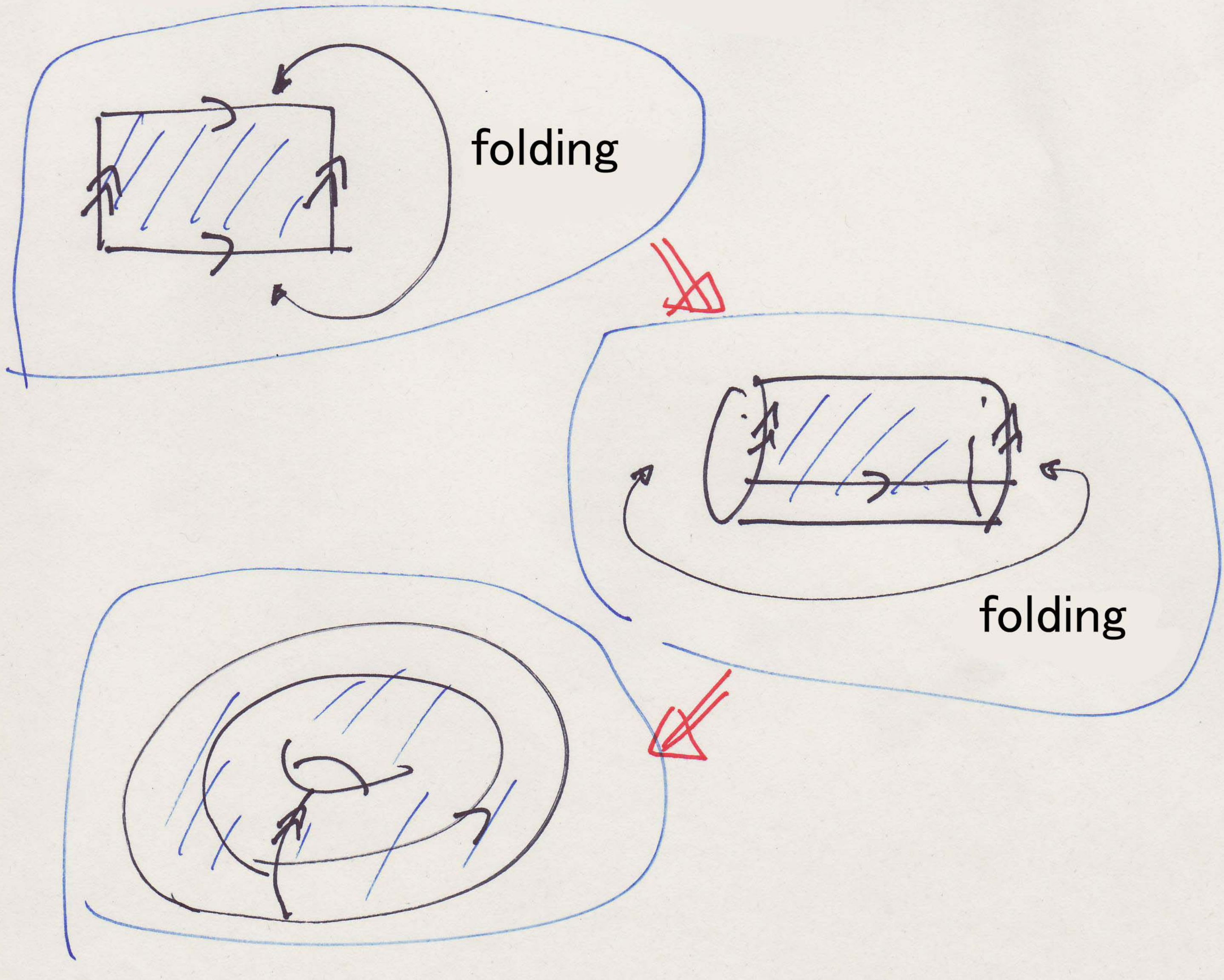
"two-dimensional sub-vector space".

(note. $Gr_2(\mathbb{R}^4)$ is known as a "four manifold".)

"Folding" in the description of torus



If a square is located in a space as at left, "folding" can be realized by elongating and contracting it as shown below.



- When a square is located in the space, $(\mathbb{R}^3)$, "folding" is physically possible by elongating and contracting it like a rubber.

However,

- for considering $\{(x+n,\ y+m)\}$ as a single "point", the space $(\mathbb{R}^3)$ is not needed.

- Grassman variety did not have surroundings originally.

$\Rightarrow$ Need for abstract and free concepts of

$\begin{cases} \text{topologic space} \\ \text{manifold} \end{cases}$

What is manifold?

n-manifold

A topologic space*, and

a space whose points are

homeomorphic to around the origin of $\mathbb{R}^n$,

the n-dimensional

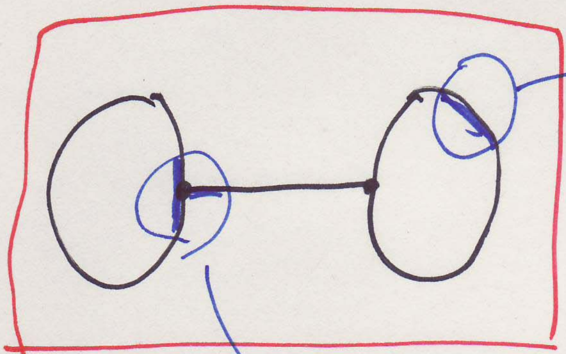
Euclidean space.

unexplained

An example of non-1-manifold

homeomorphic

to surround of $\mathbb{R}^1$'s origin

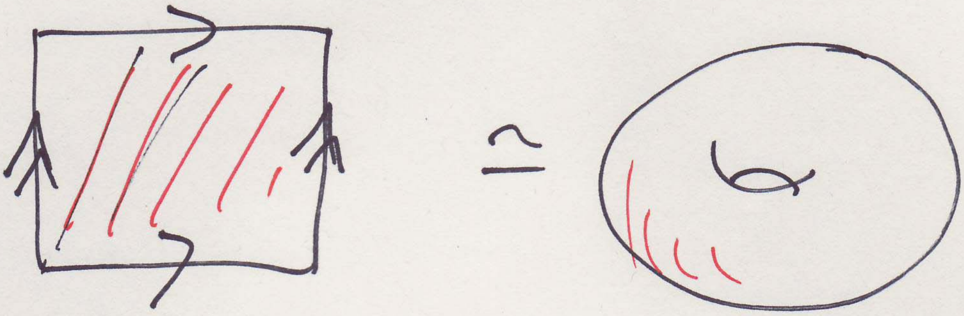


not

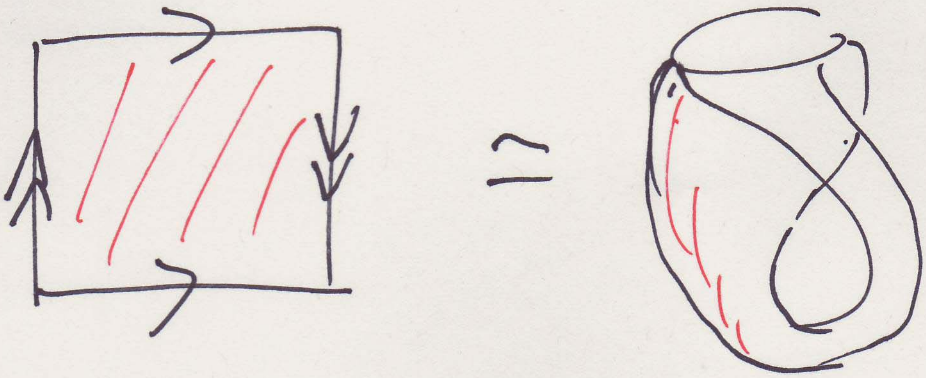
homeomorphic to $\mathbb{R}^1$'s origin

(* Hausdorff paracompact topologic space)

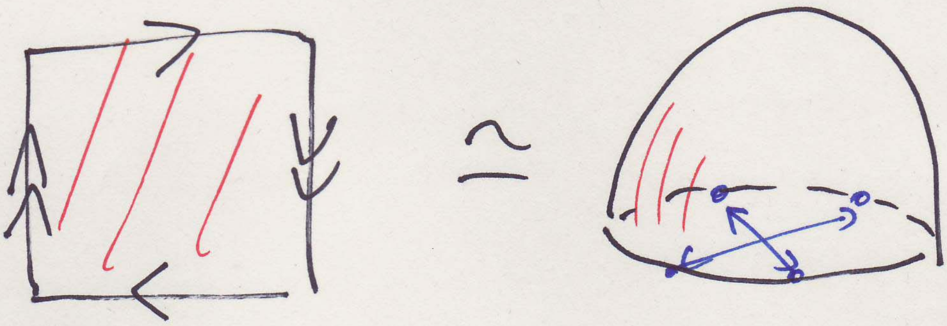
Examples of 2-manifold



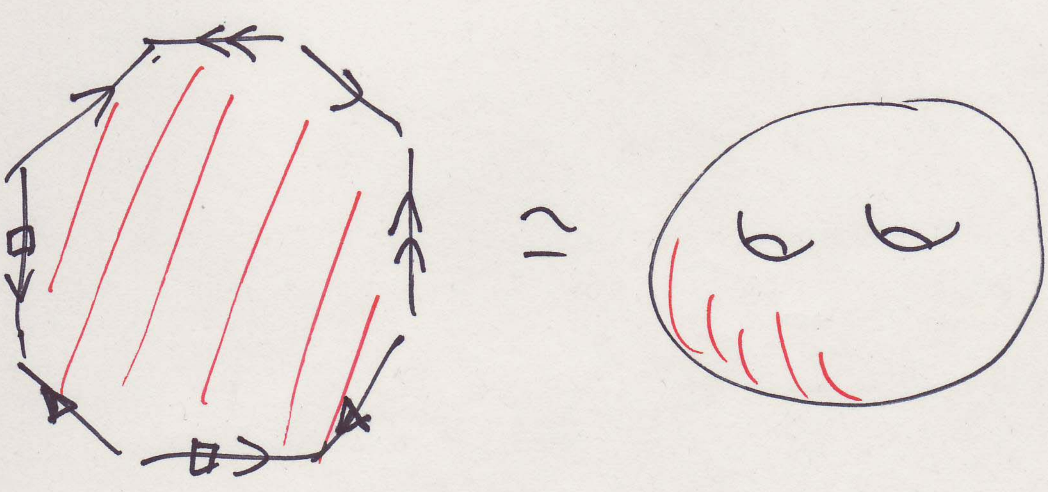
torus



Klein bottle



projected plane



Topologic space

When

the set X is a topologic space, for each

X 's subset A, intuitively, the set of all the points on A and A 's border

named $\overline{A}$ is attached, and

the correspondence $A \longmapsto \overline{A}$ satisfies the axiom* below.

$$\begin{cases} ① & \overline{\phi} = \phi \\ ② & \overline{A \cup B} = \overline{A} \cup \overline{B} \\ ③ & \overline{A} \supset A \\ ④ & \overline{\overline{A}} = \overline{A} \end{cases}$$

(* Various equivalent axioms. Such as median classes.)

Topologic spaces are dealt with not only in geometry, but also in many fields such as algebra, number theory and analysis, and they are very important.

In modern mathematics, the concept of topologic spaces is essential.

Just as the concept of vector spaces is important.

The final page of

A Forget the surroundings and focus on the spaces themselves!

$$\text{I} \quad \mathbb{R}^3 \text{ introduced with}$$

$$\boxed{(\Delta x)^2 + (\Delta y)^2 - (\Delta z)^2}$$

!!

Looking back ...

normal distance

$$(\text{distance})^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

Abbreviate this to...

$$\boxed{ds^2 = dx^2 + dy^2 + dz^2}$$

quantitation

mock distance

$$\text{"}(\text{mock distance})^2\text{"} = \underbrace{(\Delta x)^2 + (\Delta y)^2 - (\Delta z)^2}$$

Abbreviate this to...

$$\boxed{ds^2 = dx^2 + dy^2 - dz^2}$$

In $\mathbb{R}^3$, it might be negative.

Does

quantitation

$\boxed{\text{mock distance}}$ seem like an unreality

extraneous to a real distance?

However, according to

Einstein's special theory of relativity,

For
- time t (unit: second)
- space coordinates x_1, x_2, x_3 (unit: light-second)

$$\boxed{dx_1^2 + dx_2^2 + dx_3^2 - dt^2}$$

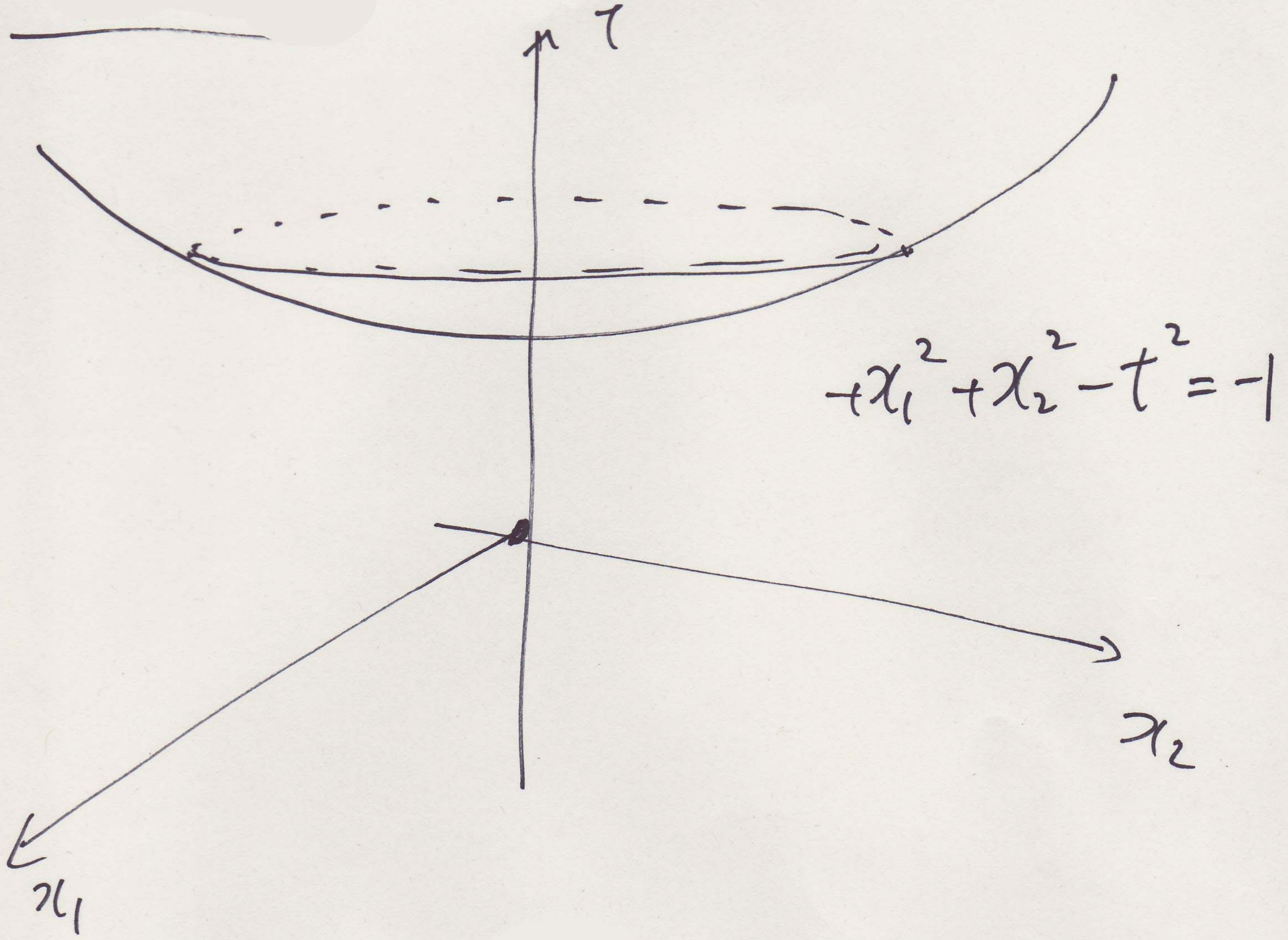
is physically meaningful.

($\sqrt{-(dx_1^2 + dx_2^2 + dx_3^2 - dt^2)}$

calculates elapse of proper time of

an object.)

That means...



At $t=0$, start from the origin and proceed toward various directions.

A hyperboloid like the one shown above is obtained if " a time and a location of the moving object after each second in proper time passes" is plotted.

($x_3=0$ in the graph above.)

If x_3 is written, too,

$$x_1^2 + x_2^2 + x_3^2 - c^2 t^2 = -1$$

light−velocity

A physical, genuine distance

in this "three-dimensional space"

(under a theory of special relativity)

is nothing but the distance of non−Euclidean geometry.

In this sense, we can say that

non−Euclidean geometry exists as

spherical geometry.

The end of B

Summary

A : Depart from model,
examine a space itself, and
consider many possibilities freely

B : Sometimes, the products of these examples of free thinking are unexpectedly effective in understanding the reality beyond the bounds of imagination.

In the next and succeeding lectures, let's take a close look at those products of free thinking

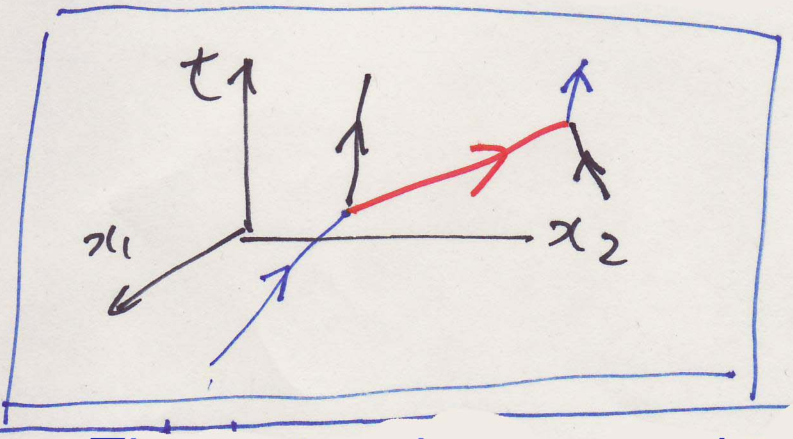
Appendix

"Dynamical Definitions" of the Theory of Special Relativity and Time

Suppose that in time-space,* various kinds of particles** are flying around. They are splitting or bumping into each other.

When the distance between them becomes greater than a certain range, no "power" is exerted.

Let us assume that "a law of conservation" was discovered as an analysis of the video recording actions of these particles.



* Time-space is supposed to be affine.

** Existence of "particles of a same kind" is used used in the following discussion.

Law of Conservation

① Every moving particle has

a vector $\begin{pmatrix} m \\ P_1 \\ P_2 \\ P_3 \end{pmatrix}$ along the locus of time space.

② When they impact and break up,

the sums of these vectors are conserved.

Q. Is the law of conservation shown above meaningful?

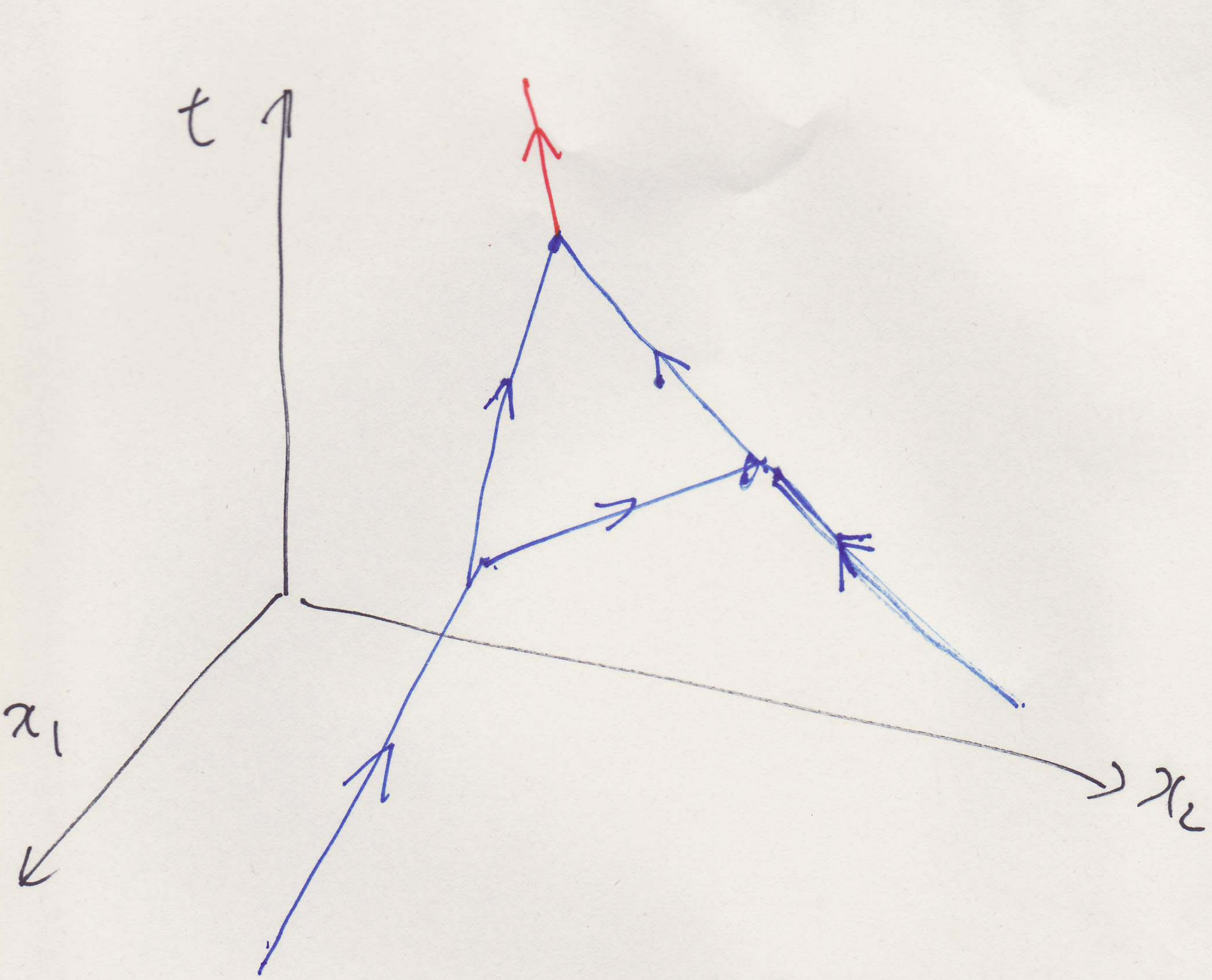
In other words, is it falsifiable by experiments?

A. Yes. " A prediction" can be made using

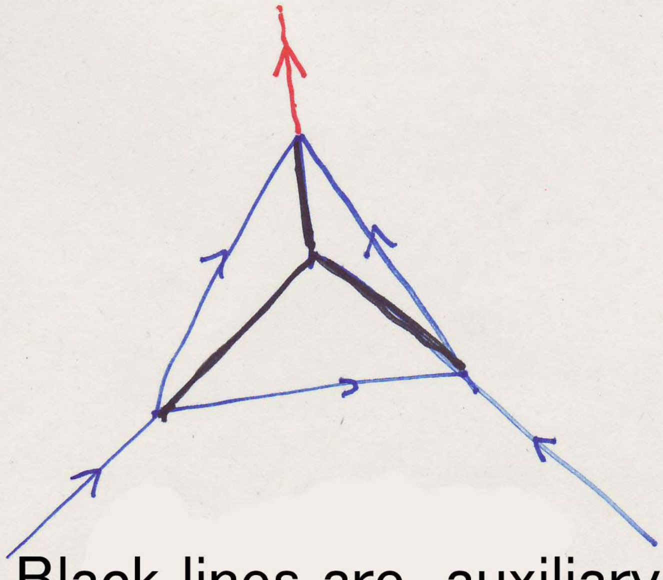
the law of conservation above.

You just have to check if this prediction

is true or not.



If the blue locus is known, the red locus can be predicted as follows.

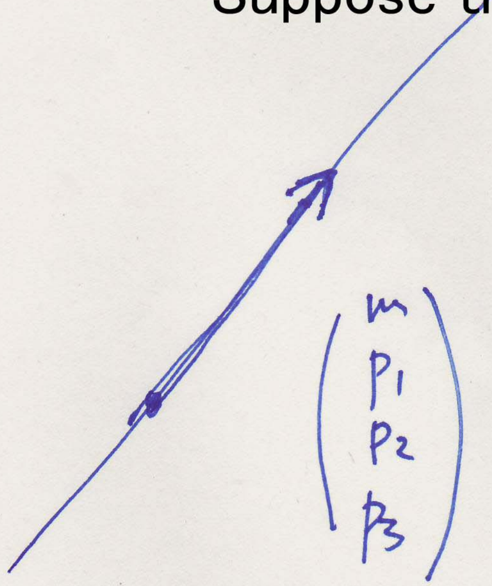


Black lines are auxiliary lines.

- Suppose that this law of consevation is true.
- Then, this vector $\begin{pmatrix} m \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$ has a “dynamic meaning” defined by this law.

  *

- Suppose that “a particle of a certain kind”** is moving toward various directions. Using this particle, define proper time as shown below.



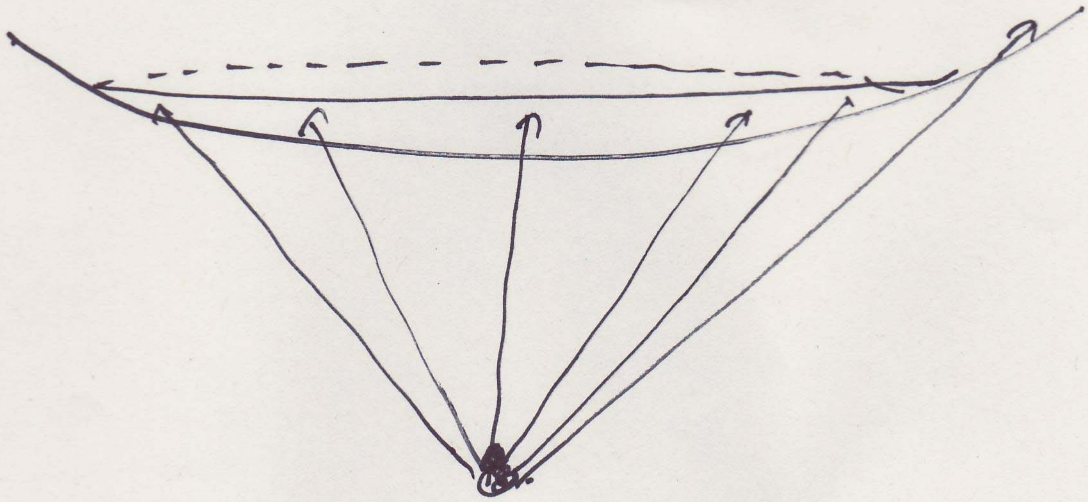
$\begin{pmatrix} m \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$

<u>definition</u>	Define the measurement of proper time that the length of the vector is equal to elapse of proper time = 1 .

---

* overall constant is unstable.

** something like an electron, for example



Release particles from a point in time-space to many directions, and plot (time, location) at the elapse of proper time =1. (As shown in the illustration above)

Under the assumption of adequate direction, it can be proved mathematically that a curve above is a quadric surface.

When

reference ·· $\vec{V}_{-1}, \vec{V}_0, \vec{V}_1$ -- satisfy $\vec{V}_k = \lambda(\vec{V}_{k-1} + \vec{V}_{k+1})$

terminal points of these vectors are on a single quadratic curve.

When this quadric surface is a hyperboloid, (under the assumption of adequate isotropy and proper coordinates) the equation can be written as

a regular number (light velocity)

$$\boxed{x_1^2 + x_2^2 + x_3^2 - c^2 t^2 = -c^2 t_0^2}$$

(t_0 = unit time)

Looking back on the definition of proper value, the equation above is equal to:

$$\boxed{p_1^2 + p_2^2 + p_3^2 - c^2 m^2 = -c^2 m_0^2}$$

(m_0 : proper value for a particle)
(static mass)

The former is essential for a geometry of special relativity theory, and the latter is essential for the dynamics of special relativity theory.

Thus, under "the dynamic definition"* of proper time, both are equal.

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* It is proved by special relativity theory that when the definition of proper time measures various physical phenomena, it shows its genuine character,

quantitation

Conclusion

- Mock distance $dx_1^2 + dx_2^2 + dx_3^2 - c^2dt^2$

of time space is in a relationship neither too close to nor too remote from phenomena in time space such as "the law of conservation".

- In geometry in pure mathematics, in a space where a mock distance was not clearly defined, sometimes,

quantitation a mock distance is introduced.

- We may say that this mock distance is showing us something primordial about a subject space then.