Analysis and Multidiscipline in Geometry

— Duality and Self in a Mirror —

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The previous lecture

A $\boxed{\text{a free use of}}$

imagination, conceptual power,

theoretical possibility

– release from "reality/practical use of geometry"

B When reality goes over imagination,

A is sometimes useful

A	B
Model of non-Euclidean geometry	Geometry of special relativity theory
Riemann geometry	special relativity theory

(reference: "Einstein's Lectures at Komaba"
University of Tokyo Press)

Appendix

Momentum to a concept of "a space itself" without surroundings → topology, manifold etc.

① Non-Euclidean geometry

(Lobachevsky, Bolyai, Gauss)

② Riemann geometry (← Gauss's theorem of curves

(Gauss, Riemann)

- Historically, ① was less popular.
- Especially, ② 's notion of "curvature"

reference: "*Bernhard Riemann* 1826–1866 "
D. Laugwitz , translation by A. Yamamoto
Springer Verlag Tokyo ,1999

Encounter Between Concepts

☆ Sometimes, a concept or a viewpoint of a certain field has/can unexpectedly have a practical role in other fields.

- Math and Physics

  (ex. A and B in the previous slide, a role of complex numbers in quantum mechanics)

- In physics

  example: (formal similarity of statistical mechanics and quantum field theory, the concept of conservative quantity etc.)

- In math

  (similarity of prime numbers and geodesic lines etc.)
  (T. Sunada)

⋮

cf Sometimes, math meets "practicality".

e.g. number theory and cryptography (c.f. Katsura)

Today

An introduction to the cross-conceptual mathematical notion used in algebra, geometry, and analytics.

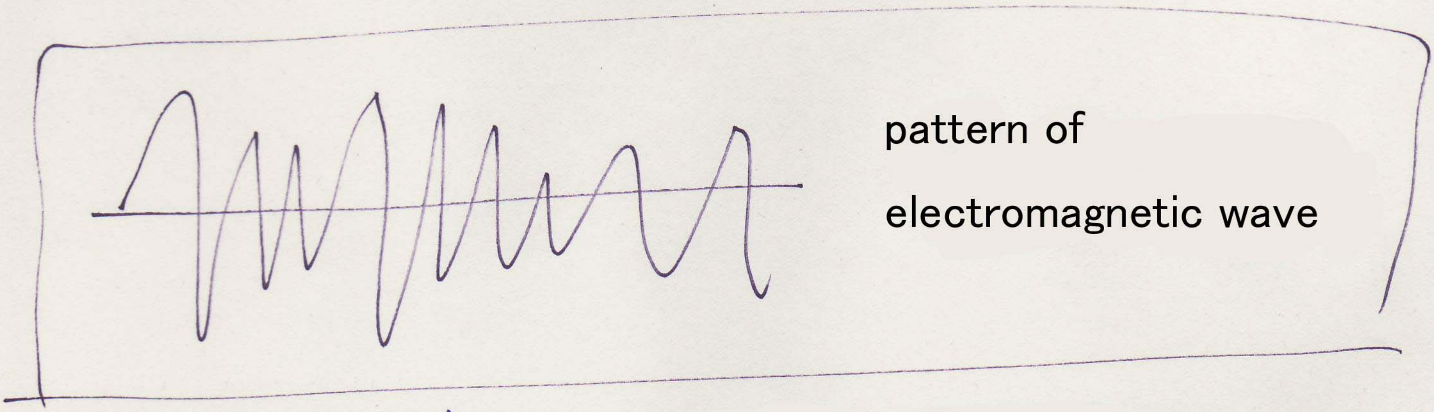
$\boxed{\text{duality}}$

Let us look into how it appears especially in manifold geometry.

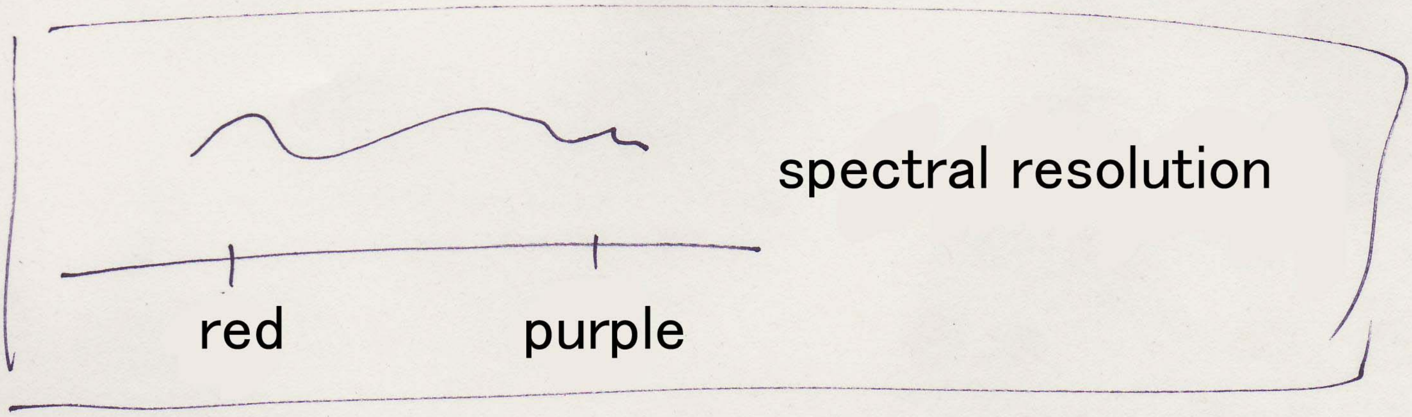
- Poincare duality
- Tangent vector and cotangent vector
- de Rham's theorem

What is duality?

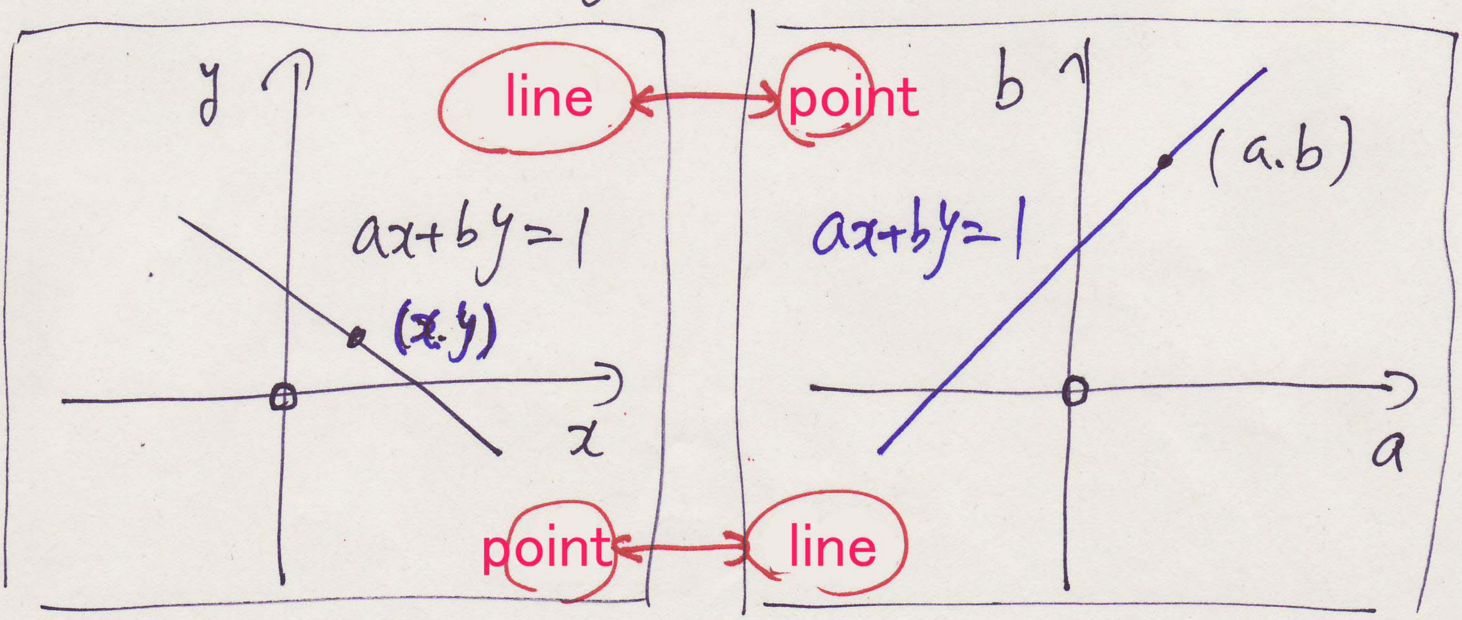
(1) Two expressions of a light wave



$\Downarrow$ Fourier transform ,Fourier development



(2) $ax + by = 1$ 2 analyses



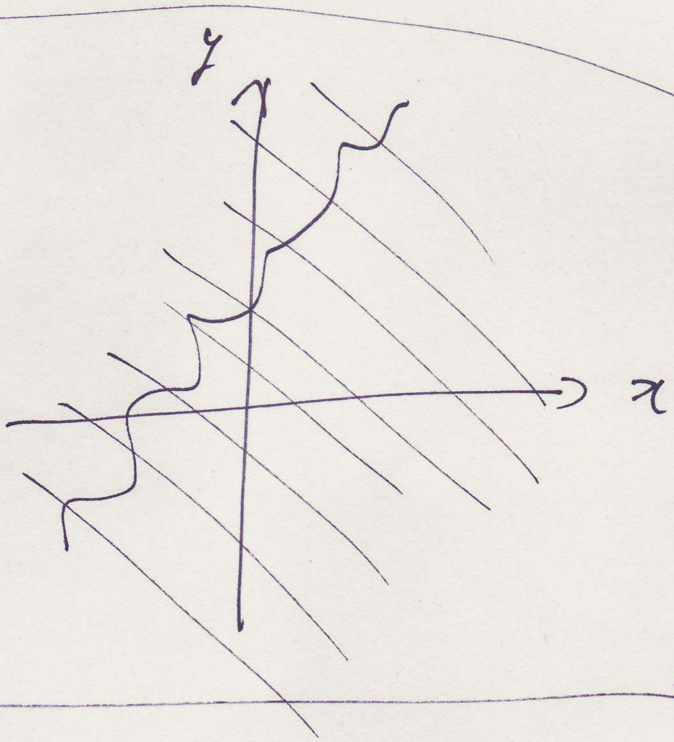
$\boxed{(1)\ \text{spectral resolution} \Longleftrightarrow (2)\ \text{line} \longleftrightarrow \text{point}}$

$$\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$$(a, b) \qquad \begin{pmatrix} x \\ y \end{pmatrix} \qquad ax+by$$

$$e^{\sqrt{-1}\,(ax+by)}$$

plane wave



xy plane function

$\updownarrow$ expressed by superposition of plane waves

ab plane function

This superposition gives function of a, b.

Little generalization of dot product

- $C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \end{pmatrix}$ fix

- $\vec{a} = (a, b, c)$

- $\vec{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$

$$\boxed{\begin{array}{ccccc} \mathbb{R}^3 & \times & \mathbb{R}^4 & \xrightarrow{\text{bilinear}} & \mathbb{R} \\ \vec{a} & & \vec{x} & & \vec{a}\,C\,\vec{x} \end{array}}$$

Consider $\forall \vec{a}$ with a focus on its effect upon $\vec{x}$

Consider $\forall \vec{x}$ with a focus on its effect upon $\vec{a}$

$\mathbb{R}^3 \ni \vec{a}, \vec{a}'$

Let us call this equivalence relation $\underset{\mathbb{R}^3}{\sim}$

$\vec{a} \underset{\mathbb{R}^3}{\sim} \vec{a}' \iff$ For all $\vec{x} \in \mathbb{R}^4$,

$$\vec{a} \, c \, \vec{x} = \vec{a}' \, c \, \vec{x}$$

When

$\vec{a} \underset{\mathbb{R}^3}{\sim} \vec{a}'$, and $\vec{a}$ & $\vec{a}'$ are identified,

the set obtained is written as $\mathbb{R}^3 / \underset{\mathbb{R}^3}{\sim}$

$\mathbb{R}^4 \ni \vec{x}, \vec{x}'$

Let us call this equivalence relation $\underset{\mathbb{R}^4}{\sim}$

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When

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the set obtained is written as $\mathbb{R}^4 / \underset{\mathbb{R}^4}{\sim}$

$$\mathbb{R}^3/_{\sim_{\mathbb{R}^3}} \times \mathbb{R}^4/_{\sim_{\mathbb{R}^4}} \longrightarrow \mathbb{R}$$

$$[\vec{a}] \qquad [\vec{x}] \qquad \vec{a}\, C\, \vec{x}$$

Then, both $\mathbb{R}^3/_{\sim_{\mathbb{R}^3}}$ and $\mathbb{R}^4/_{\sim_{\mathbb{R}^4}}$ can perfectly recognize themselves by seeing themselves in a mirror.

no degeneracy

FACT

$$\boxed{\dim_{\mathbb{R}} \mathbb{R}^3/_{\sim_{\mathbb{R}^3}} = \dim_{\mathbb{R}} \mathbb{R}^4/_{\sim_{\mathbb{R}^4}}}$$

vector spaces

Discrete Version

$$\mathbb{Z}^2 \times \mathbb{Z}^2 \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

$$(a.b) \qquad \begin{pmatrix} x \\ y \end{pmatrix} \qquad ax+by \bmod 2$$

Q. For $\begin{cases} \vec{a} = (a.b) \\ \vec{a}' = (a'.b') \end{cases}$, assume that

all $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ satisfy

$$\vec{a} \cdot \vec{x} \bmod 2 = \vec{a}' \cdot \vec{x} \bmod 2$$

Then, what is the relationship between $\vec{a}$ and $\vec{a}'$?

A. $\begin{cases} a \equiv a' \bmod 2 \\ b \equiv b' \bmod 2 \end{cases}$

This question is asking , in other words,

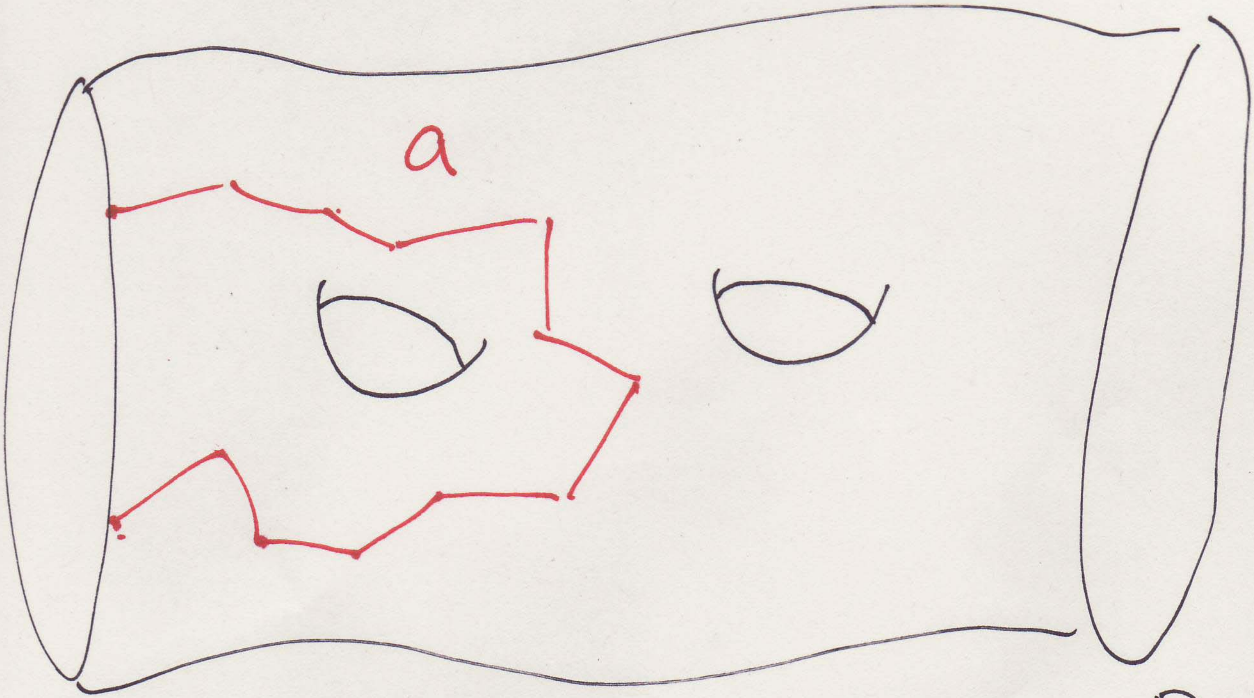
「How much of themselves is understood by looking in a mirror?」

From here, let us consider

the similar problem in geometry.

One-dimensional chain on a curve

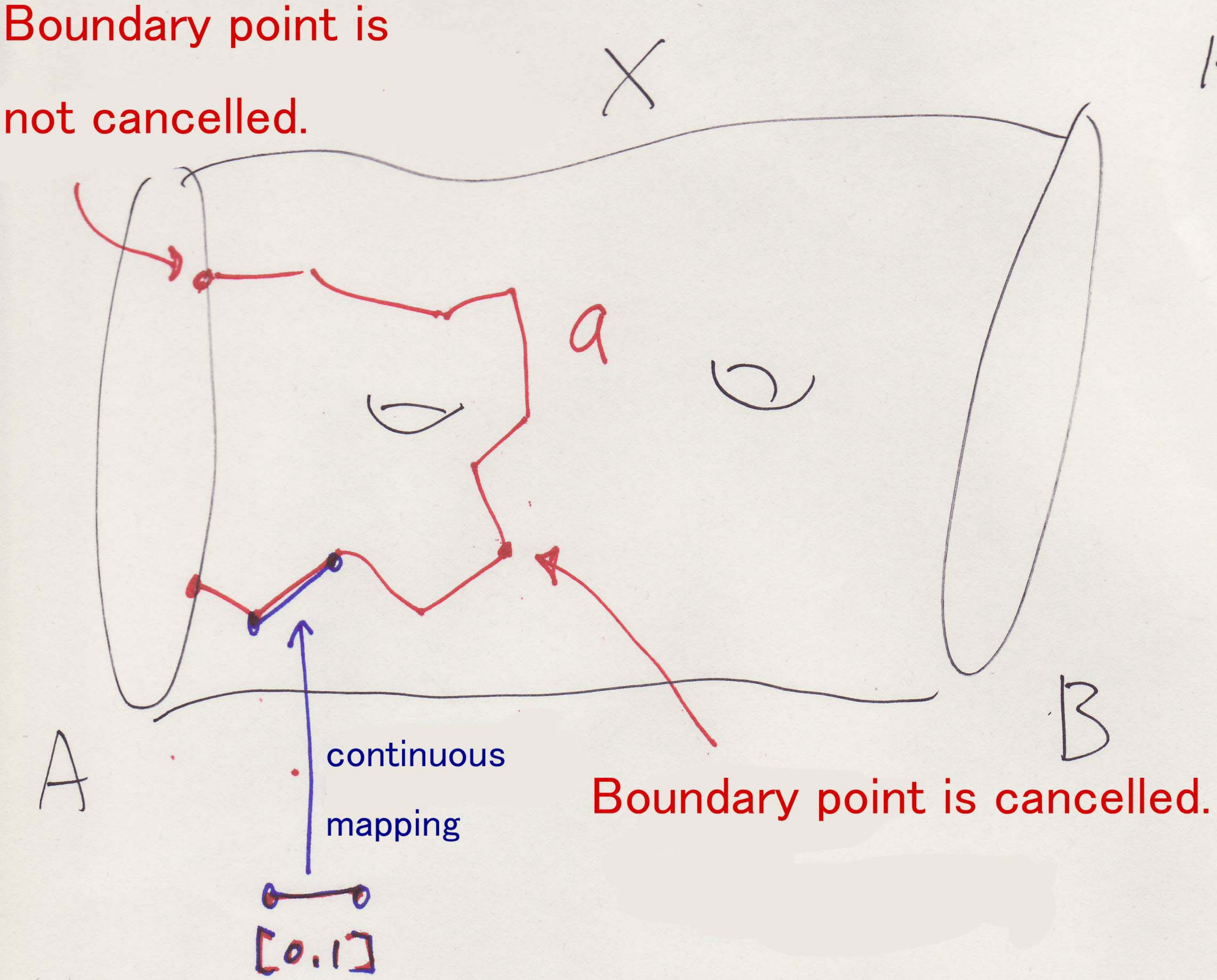
Consider the figure below.



A (part of boundary) X (curve) B (part of boundary)

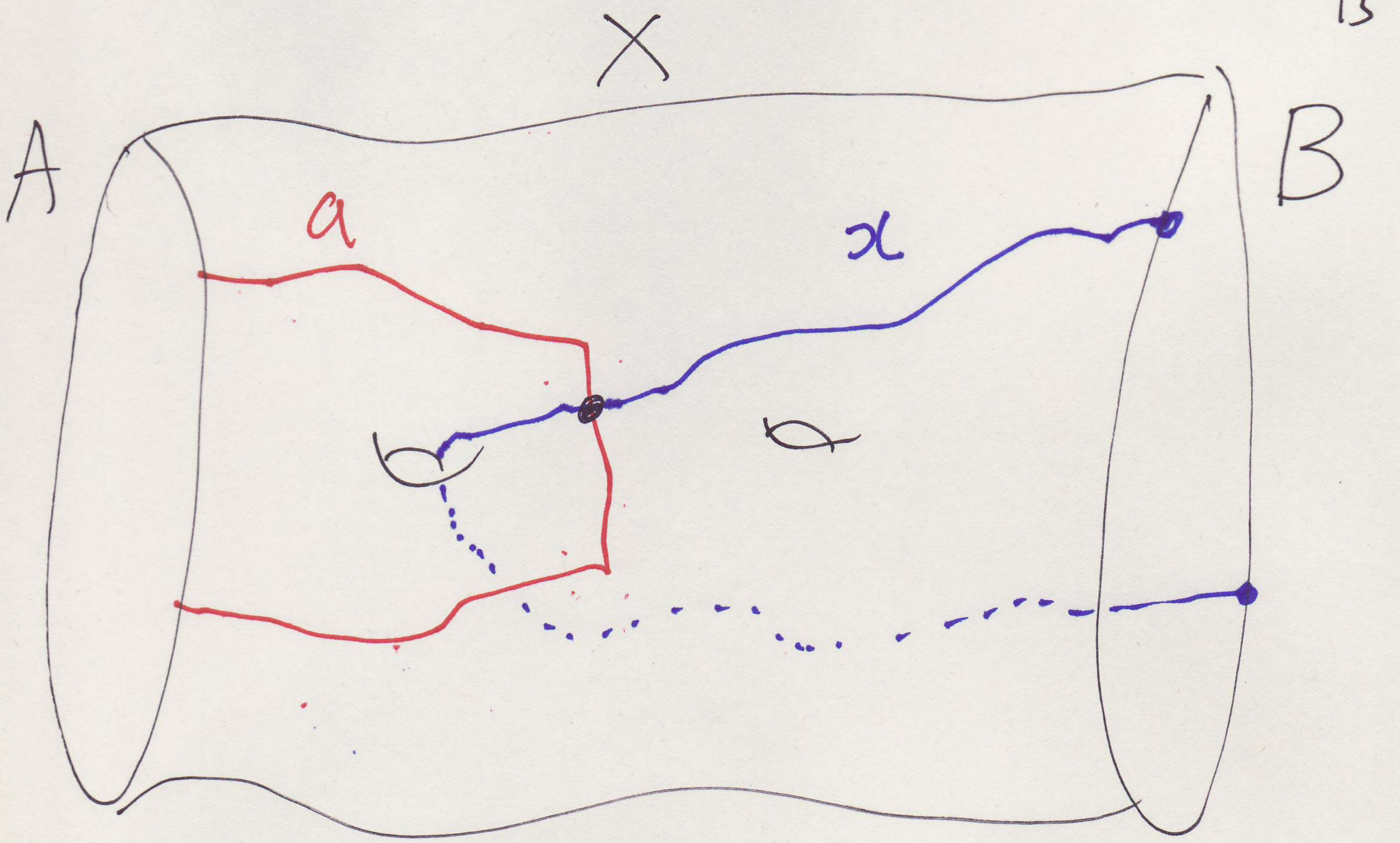
$$\begin{cases} \cdot\ \dim X = 2 \\ \cdot\ \partial X = A \cup B \quad (A \cap B = \phi) \end{cases}$$

X Two-dimensional compact manifold



a : One−dimensional chain finite number of continuous mapping images gathered from $[0,1]$

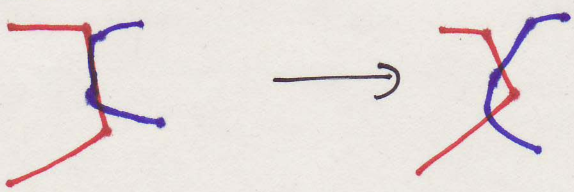
$$Z_1(X, A) \underset{\text{definition}}{:=} \left\{ a \;\middle|\; \begin{array}{l} \text{One−dimensional chain} \\ \text{whose boudary points} \\ \text{that are not cancelled are} \\ \text{all on A} \end{array} \right\}$$



$$Z_1(X,A) \times Z_1(X,B) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

$$a \qquad\qquad x \qquad\qquad \#(a \cap x) \bmod 2$$

Consider whether the number of points at the intersection of a with x is even or odd.



Shift slightly, and count intersections “cross-sectorally”.

What can be understood from an image in a mirror?

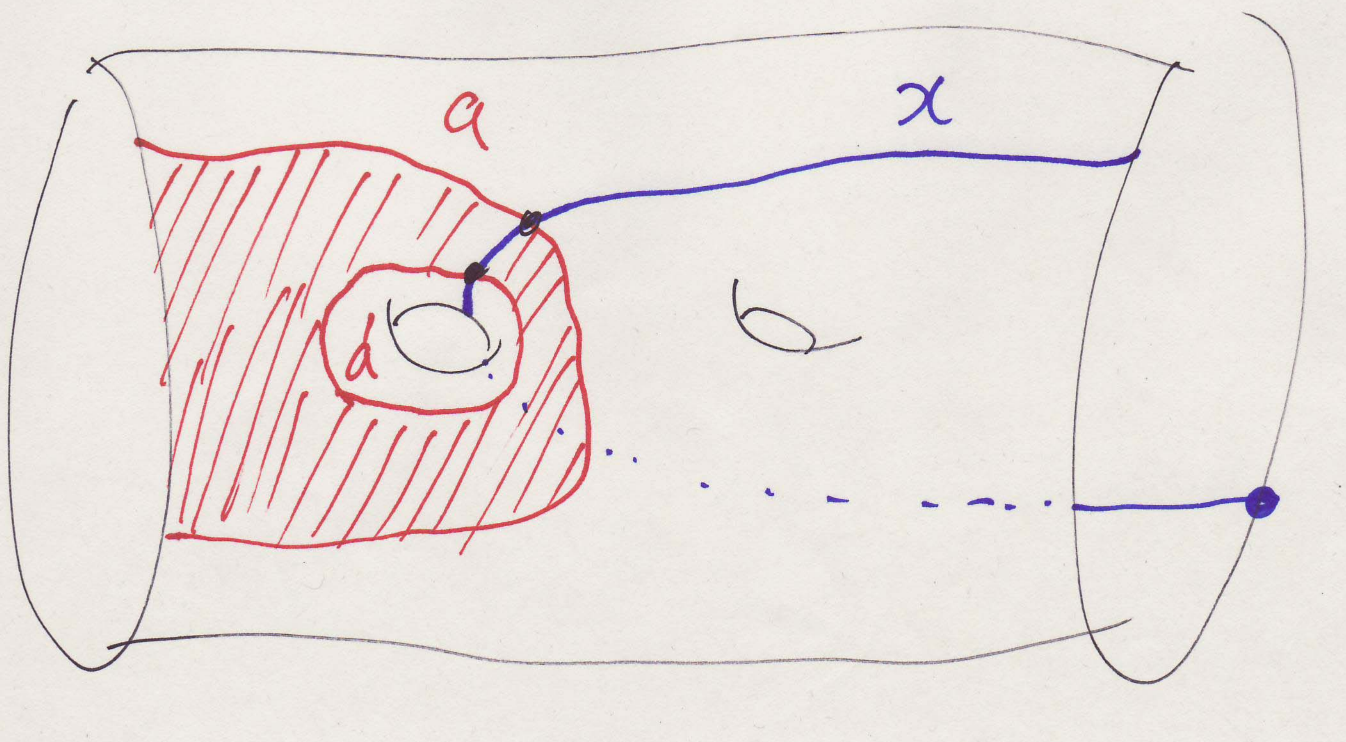
Q. When $a, a' \in Z_1(X, A)$ satisfies,

for all $x \in Z_1(X, B)$,

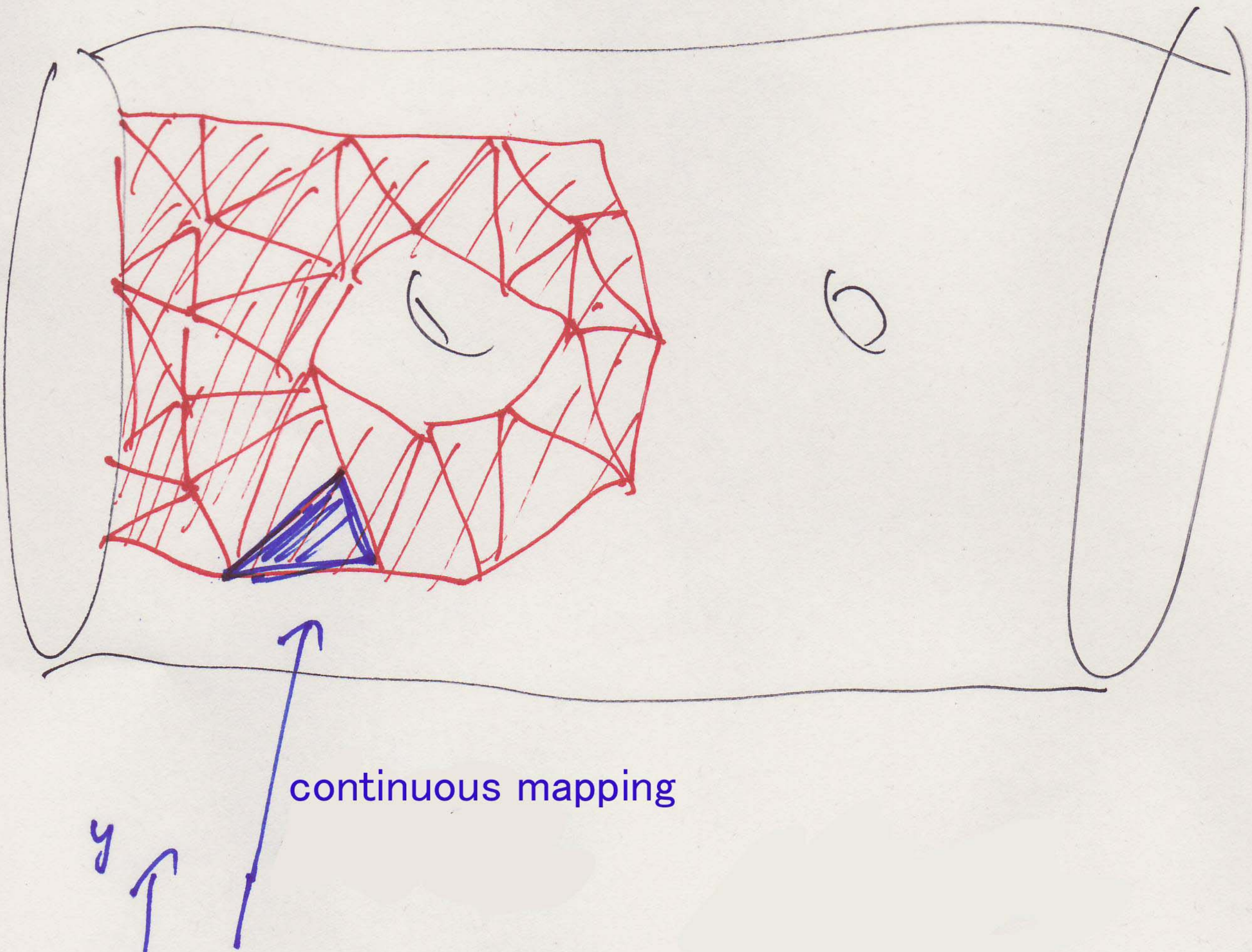
$$\#(a \cap x) \equiv \#(a' \cap x) \mod 2$$

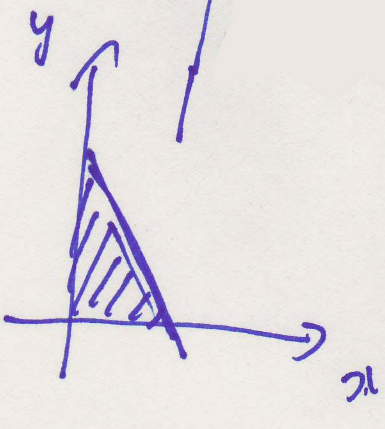
what is the relationship between a & a' ?

Examination The above condition is true when "a membrane can be formed between a and a'.



When can a membrane be formed ?





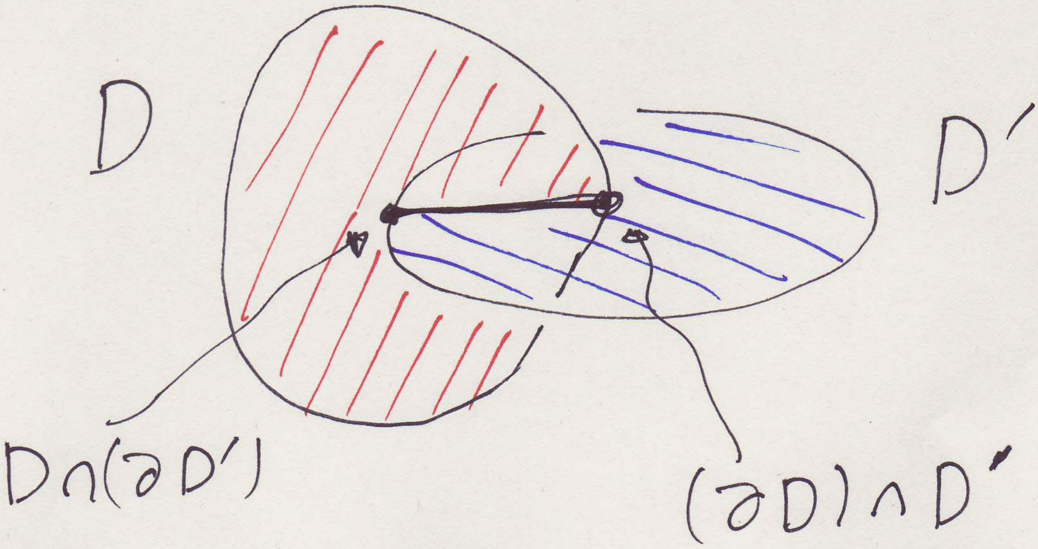
$$\left\{ \begin{array}{l} 0 \leq x, y \\ x + y \leq 1 \end{array} \right\}$$

A finite number of continuous mapping images are gathered from the triangle.

When images whose destination of boundary sides cancel each other are excluded, $a \cup a'$ is satisfied or they are included in A.

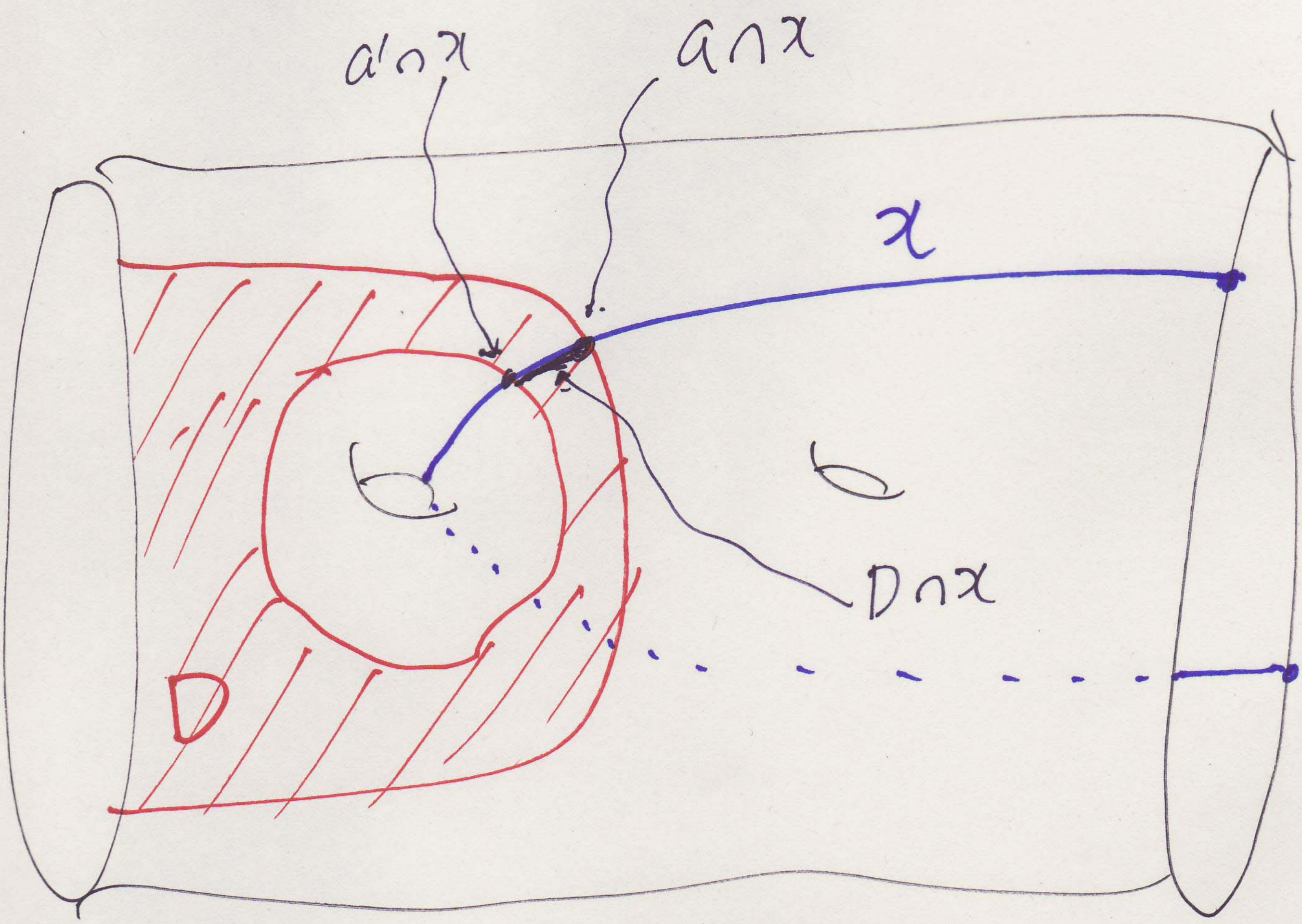
Why?

$$\partial(D \cap D') = \{(\partial D) \cap D'\} \cup \{D \cap \partial D'\}$$



If this is applied to

$$\begin{cases} D = \text{a membrane between } a \,\&\, a' \\ D' = x \end{cases}$$



$$\partial(\underbrace{D \cap x}) = \underbrace{(\partial D \cap x)}_{(a \cap x) \cup (a' \cap x)} \cup \underbrace{(D \cap \partial x)}_{\text{no intersection } *}$$

one-dimensional line

Number of boundary points on one-dimensional line is always even.

$$\therefore \ \#(a \cap x) \equiv \#(a' \cap x) \mod 2$$

$*$ D, a, a' can be moved slightly not to cross with B.

Poincare duality

establishment of the opposite

Theorem (Poincare)

When $a, a' \in Z_1(X, A)$ satisfies,

for all $x \in Z_1(X, B)$,

$$\#(a \cap x) \equiv \#(a' \cap x) \mod 2$$

it is possible to form a membrane between a and a'.

Number of intersections can tell us whether a membrane exists or not.

Strategies for proof

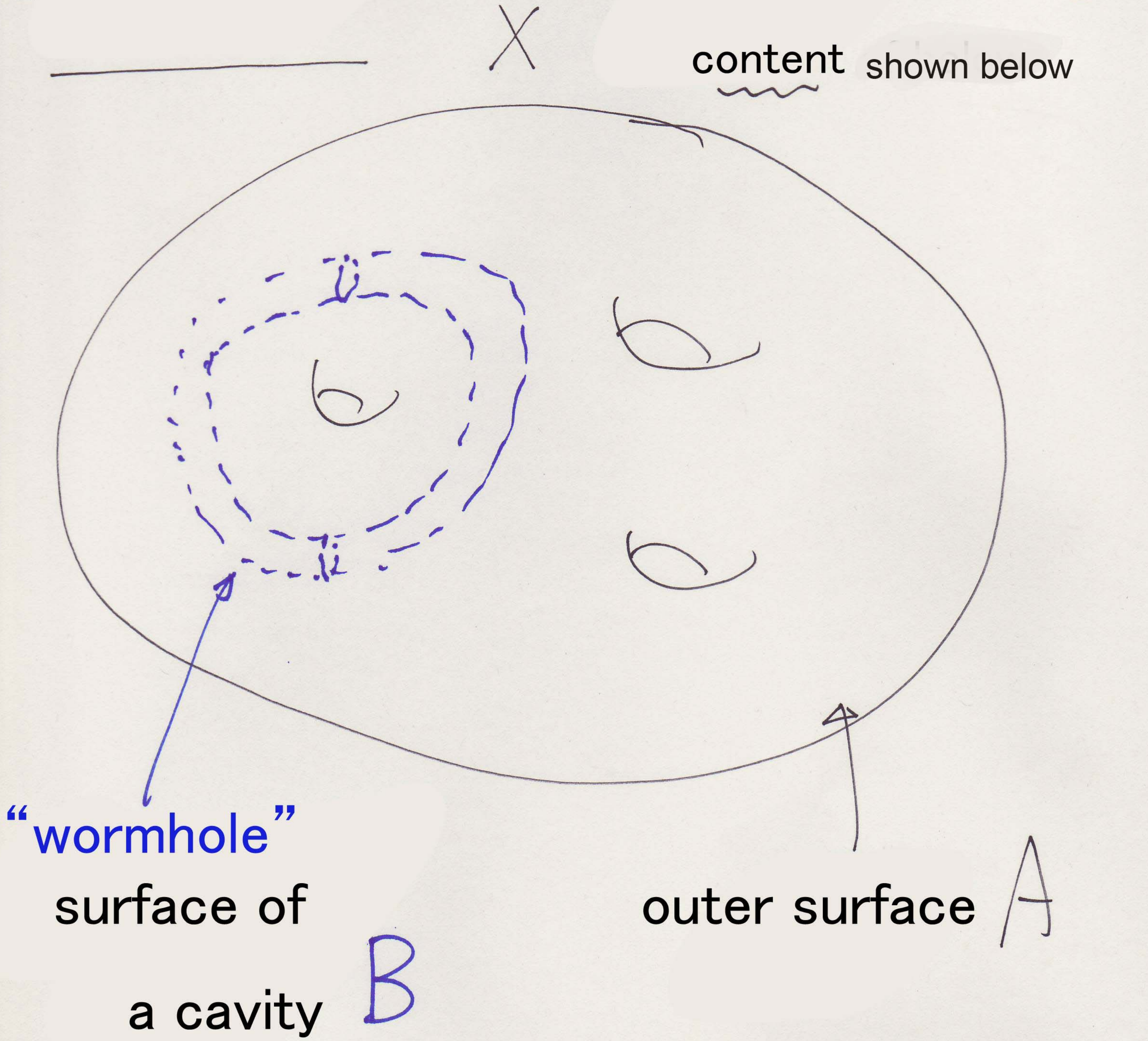
…Many strategies are possible.

(1) Use inductive method concerned with complexity of the manifold X

(2) Use "Morse theory"

(3) . . . . .

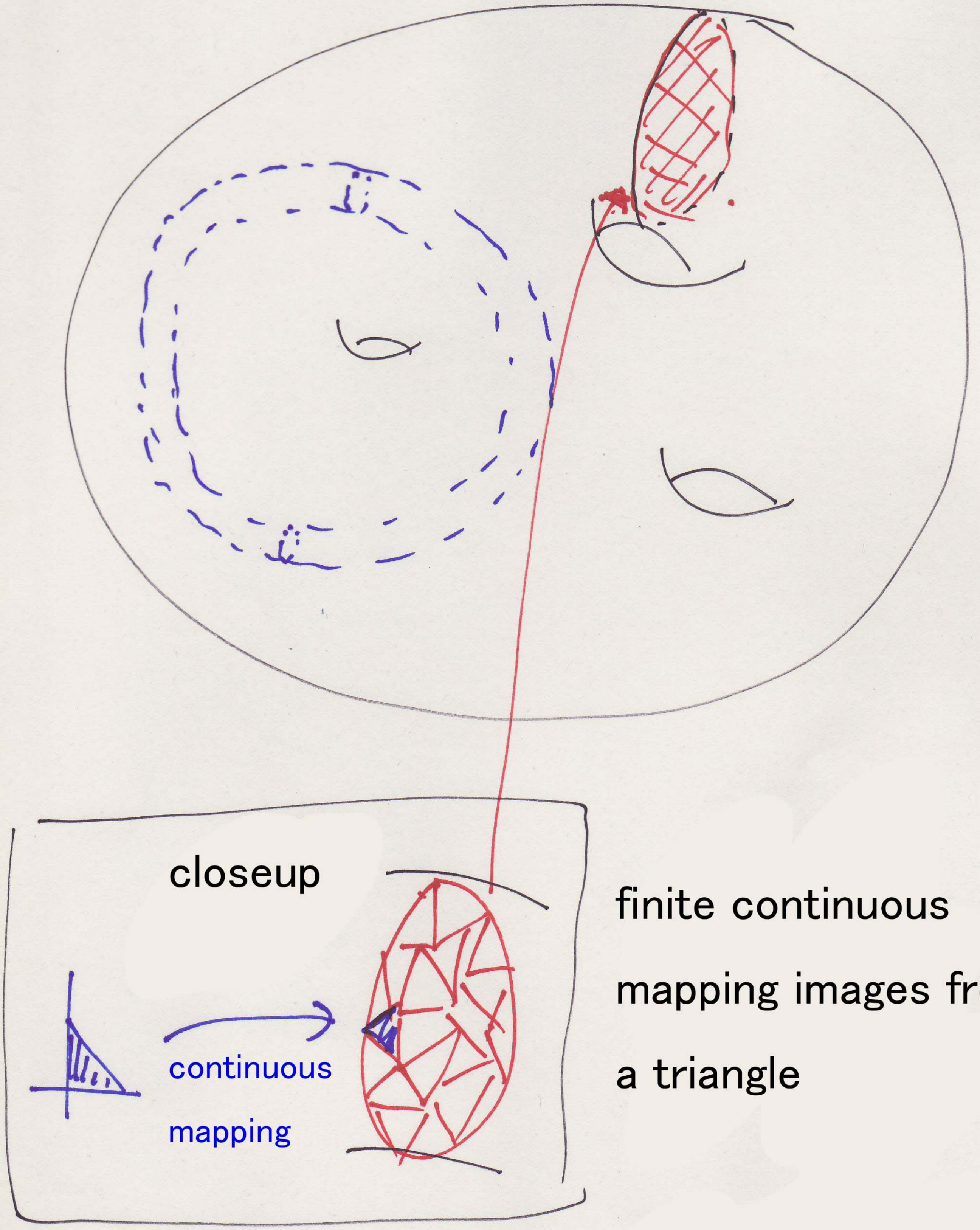
Higher dimensional version



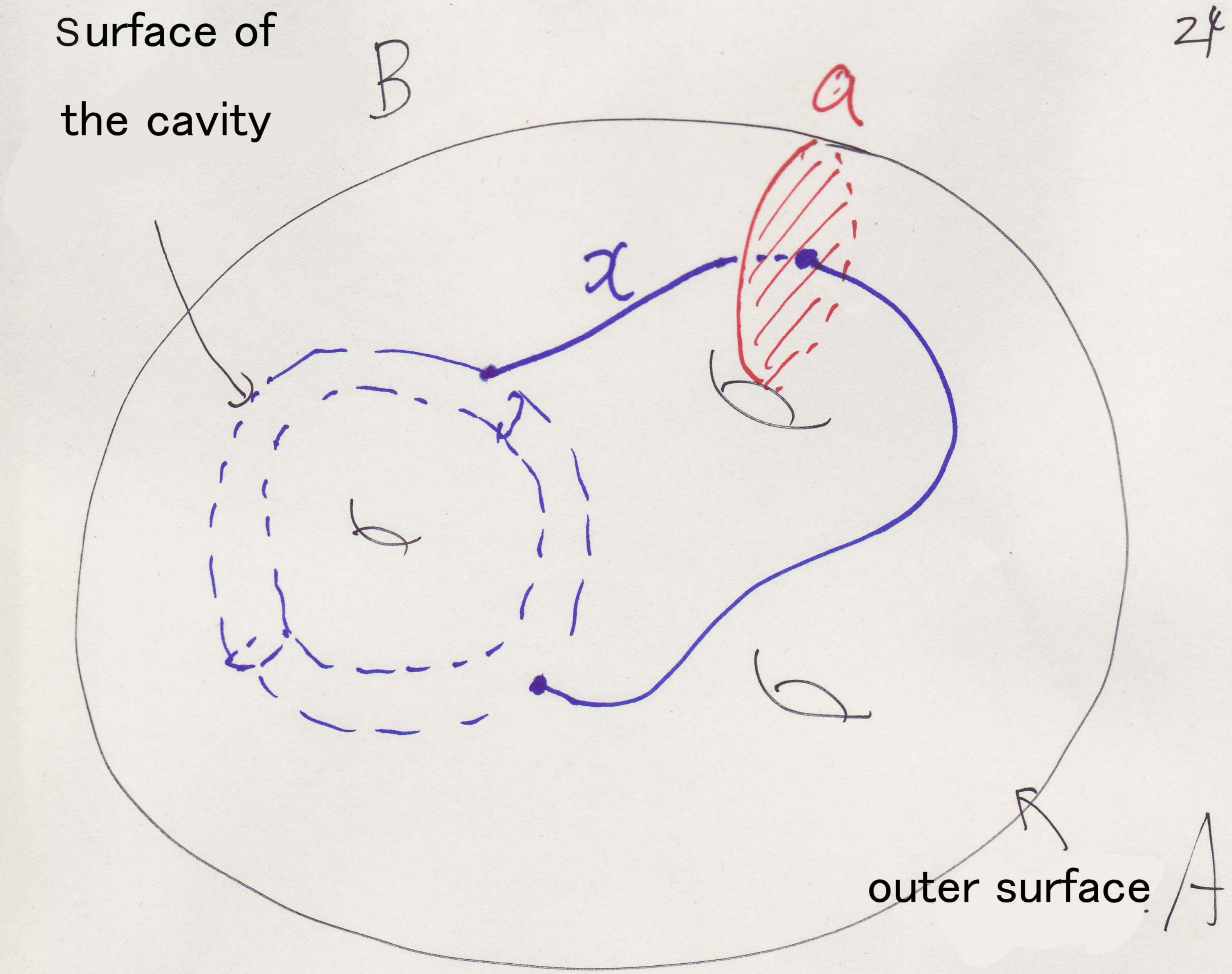
$\dim X = 3$

$\partial X = A \cup B \qquad (A \cap B = \phi)$

Two-dimensional chain



$$Z_2(X, A) = \left\{ \begin{array}{l} \text{Two-dimensional chain of } X \\ \text{whose boundary is all on } A \\ \text{when cancelled ones are} \\ \text{excluded} \end{array} \right\}$$



$$Z_2(X.A) \times Z_1(X.B) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

$$a \qquad x \qquad \#(a \cap x) \bmod 2$$

For the above figure, $\#(a \cap x) = 1$

Poincare duality is also true under these conditions.

Generally,

X : n-dimensional compact manifold

$\partial X = A \cup B \qquad A \cap B = \phi$

$n = k + l$

$$\begin{array}{ccc} Z_k(X, A) \times Z_l(X, B) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ a \qquad\qquad x & & \#(a \cap x) \bmod 2 \end{array}$$

theorem (Poincare) When

$\#(a \cap x) \equiv \#(a' \cap x) \bmod 2$

is true for all x,

a membrane (higher dimensional version)

can be formed between a and a'.

For a better understanding of Poincare duality,

the notion of an homology group is introduced.

$$H_m(X, A; \mathbb{Z}/2\mathbb{Z}) := Z_m(X, A)/\sim$$

$a \sim a' \overset{\text{definition}}{\Longleftrightarrow}$ A membrane (higher dimensional version) can be formed between a and a'.

There is b that satisfies

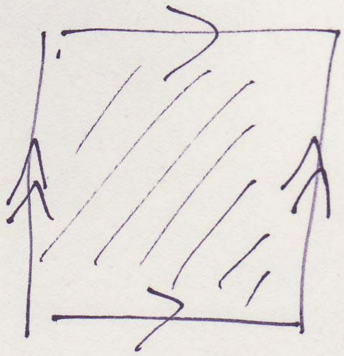
(i.e. $a \cup a' = \partial b$)

Then, $H_m(X, A; \mathbb{Z}/2\mathbb{Z})$ naturally has structure as a vectoral space on a finite body $\mathbb{Z}/2\mathbb{Z}$.

Example

Two-dimensional torus

T^2



$(\partial T^2 = \phi)$

- $H_0(T^2; \mathbb{Z}/2) = \{\, 0,\ [pt] \,\} = \mathbb{Z}/2\mathbb{Z}\,[pt]$

- $H_1(T^2; \mathbb{Z}/2) = \{\, 0, [\rightarrow]\ [\uparrow]\ [\rightarrow\uparrow] \,\}$

  $= \mathbb{Z}/2\mathbb{Z}\,[\rightarrow] \oplus \mathbb{Z}/2\mathbb{Z}\,[\uparrow]$

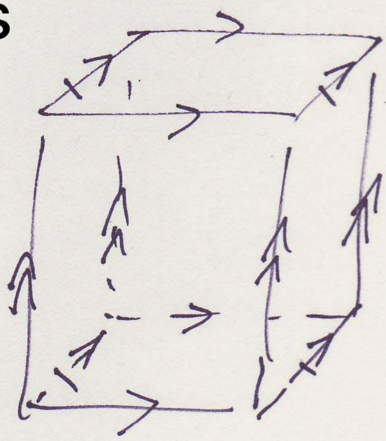
- $H_2(T^2; \mathbb{Z}/2) = \{\, 0,\ [\square] \,\}$

  $= \mathbb{Z}/2\mathbb{Z}\,[\square]$

	$H_0(T^2; \mathbb{Z}/2)$	$H_1(T^2; \mathbb{Z}/2)$	$H_2(T^2; \mathbb{Z}/2)$
#	$2 = 2^1$	$4 = 2^2$	$2 = 2^1$
$\dim_{\mathbb{Z}/2\mathbb{Z}}$	1	2	1

Three-dimensional torus

T^3



$$H_0(T^3 : \mathbb{Z}/2) = \mathbb{Z}/2\mathbb{Z}\ [pt]$$

$$H_1(T^3 : \mathbb{Z}/2) = \mathbb{Z}/2\mathbb{Z}[\rightarrow] \oplus \mathbb{Z}/2\mathbb{Z}[\uparrow] \oplus \mathbb{Z}/2\mathbb{Z}[\nearrow]$$

$$H_2(T^3 : \mathbb{Z}/2) = \mathbb{Z}/2\mathbb{Z}[\square] \oplus \mathbb{Z}/2\mathbb{Z}[\quad] \oplus \mathbb{Z}/2\mathbb{Z}[\quad]$$

$$H_3(T^3 : \mathbb{Z}/2) = \mathbb{Z}/2\mathbb{Z}[\quad]$$

	H_0	H_1	H_2	H_3
#	$2 = 2^1$	$8 = 2^3$	$8 = 2^3$	$2 = 2^1$
$\dim_{\mathbb{Z}/2\mathbb{Z}}$	1	3	3	1

In fact, generally, n-dimensional torus T^n satisfies

$$\begin{cases} \cdot\ \# H_k(T^n ; \mathbb{Z}/2) = 2^{\binom{n}{k}} \\ \cdot\ \dim_{\mathbb{Z}/2\mathbb{Z}} H_k(T^n ; \mathbb{Z}/2) = \binom{n}{k} \end{cases}$$

binomial coefficient

Especially,

when $k + l = n$

$$\boxed{\# H_k(T^n ; \mathbb{Z}/2) = \# H_l(T^n ; \mathbb{Z}/2)}$$

There is a symmetry property.

This is true for every compact manifold without boundaries.

Why??

A. It is a conclusion from Poincare duality.

$$Z_k(X, A) \times Z_\ell(X, B) \longrightarrow \mathbb{Z}/2\mathbb{Z}$$

$$\Big\downarrow$$

$$\boxed{H_k(X, A : \mathbb{Z}/2) \times H_\ell(X, B ; \mathbb{Z}/2) \longrightarrow \mathbb{Z}/2\mathbb{Z}}$$

Poincare duality means that this is an image "whose self can be perfectly understood by looking in a mirror". It also means that it is non-degeneracy

Then, from the general theory (linear algebra),

$$\dim_{\mathbb{Z}/2\mathbb{Z}} H_k(X, A : \mathbb{Z}/2) = \dim_{\mathbb{Z}/2\mathbb{Z}} H_\ell(X, B : \mathbb{Z}/2)$$

Tangent vector and cotangent vector

contravariant vector

covariant vector

$$\begin{cases} X : \text{manifold} & \text{eg. } X = \mathbb{R}^n \\ x_0 \in X \end{cases}$$

{curve} just a set

{function} can be a vector space

$$\left\{ \begin{matrix} \mathbb{R} \xrightarrow{c} X \\ 0 \longmapsto x_0 \end{matrix} \right\} \times \left\{ \begin{matrix} X \xrightarrow{f} \mathbb{R} \\ x_0 \longmapsto 0 \end{matrix} \right\} \longrightarrow \mathbb{R}$$

$$c\ ,\ f \longmapsto \left[\frac{d}{dt} f(c(t)) \right]_{t=0}$$

How these recognize each other, in other words, how much information can they get about themselves by looking in the mirror is the subject of inquiry.

$$\left\{ \begin{array}{l} \mathbb{R} \xrightarrow{c} X \\ 0 \longmapsto x_0 \end{array} \right\} \Big/ \sim \quad \times \quad \left\{ \begin{array}{l} X \xrightarrow{f} \mathbb{R} \\ x_0 \longmapsto 0 \end{array} \right\} \Big/ \sim \quad \longrightarrow \mathbb{R}$$

non-degeneracy

$$\begin{array}{ccc} \| & & \| \\ T_{x_0} X & & T^*_{x_0} X \end{array}$$

tangent vector space　　　cotangent vector space

self-understanding

by an image in the mirror

vector space

Especially,

linear

$$\boxed{T_{x_0} X = \{ \varphi : T^*_{x_0} X \longrightarrow \mathbb{R} \}}$$

Right-hand member is a structure of vector space.

Therefore, the left is a vector space.

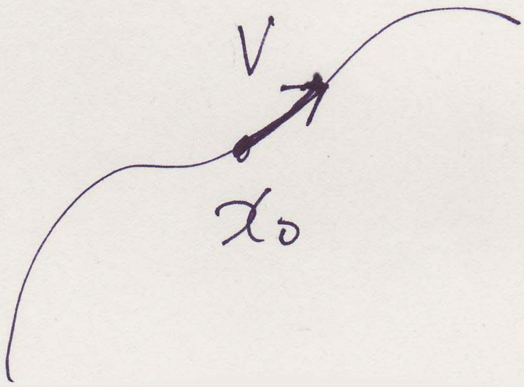
Conversely, (using this vector space structure)

$$T^*_{x_0}X = \left\{ \alpha : \underbrace{T_{x_0}X}_{\substack{\rotatebox{90}{$\in$} \\ V}} \xrightarrow{\text{linear}} \mathbb{R} \right\}$$

The direction of

x_0-crossing minimal curve

is expressed.



$T^*_{x_0}X$'s factor (cotangent vector) intuitively gives

the correspondence below.

minimal curve $\longmapsto$ minimal number

Vector field

For each point x, $T_x X$'s factor $V(x)$ is assigned, and

the correspondence $x \longmapsto V(x)$ is smooth (in a sense)

Primary differential form

For each point x, $T_x^* X$'s factor $\alpha(x)$ is assigned, and

the correspondence $x \longmapsto \alpha(x)$ is smooth (in a sense).

Original role of vector field

differential operator

$$\left[\ \left\{ \begin{array}{c} \mathbb{R} \xrightarrow{c} X \\ 0 \longmapsto x_0 \end{array} \right\} \times \left\{ \begin{array}{c} X \xrightarrow{f} \mathbb{R} \\ x_0 \longmapsto 0 \end{array} \right\} \longrightarrow \mathbb{R}\ \right]$$

vector field V function f $\longmapsto (Vf)_x$

At x_0, use $V(x_0)$ on $f - f(x_0)$

and make a number.

That means...

$$\left\{ X \xrightarrow{f} \mathbb{R} \right\} \xrightarrow{V} \left\{ X \xrightarrow{g} \mathbb{R} \right\}$$

function

$$f \longmapsto g = Vf$$

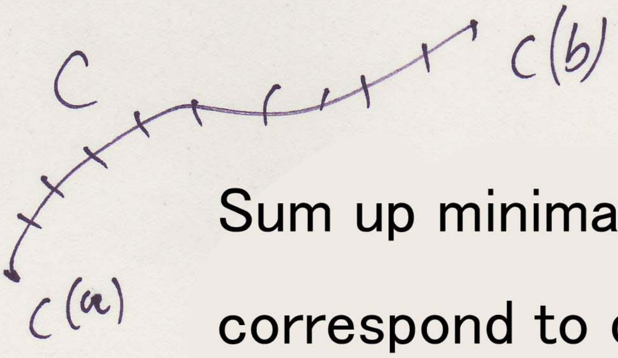
differentiation of the function f by the vector field V

Original role of primary differentiation form

What to be integrated

$$\left[\left\{ \begin{array}{c} \mathbb{R} \xrightarrow{c} X \\ 0 \longmapsto x_0 \end{array} \right\} \times \left\{ \begin{array}{c} X \xrightarrow{f} \mathbb{R} \\ x_0 \longmapsto 0 \end{array} \right\} \longrightarrow \mathbb{R} \right]$$

curve $c : [a, b] \to X$, primary differentiation form α $\longmapsto \int_c \alpha$



Sum up minimal numbers that correspond to divided minimal curve.

Primary differentiation form is to be integrated upon minimal curve. (?)

Reference

Legendre transformation

- linear programming problem
  convex programming problem

(cf Murota)

Fenschel, Morrow's duality theorem

Coon, Tucker's duality theorem

- Two descriptions in analytic dynamics

TX .. Lagrange version

T^*X .. Hamilton version

Legendre transformation

- thermal dynamics, statisticalc dynamics

par-able

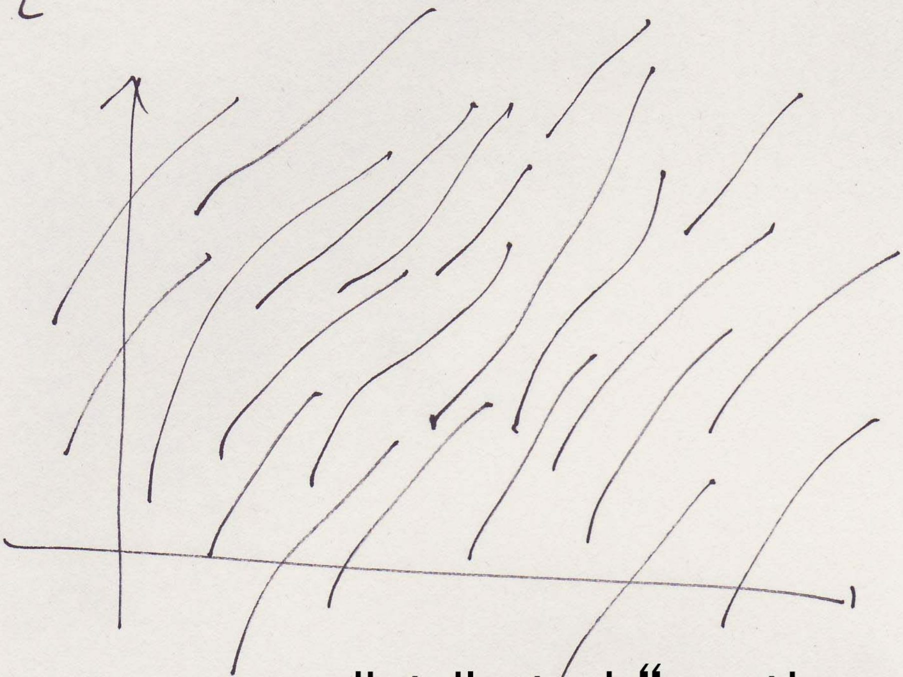
"TX" extensive variable (volume, entropy ..)

"T^*X" intensive variable (pressure, temperature _)

- math: symplectic geometry - - .

One method that gives a first differential form on a plane

$X = \mathbb{R}^2$

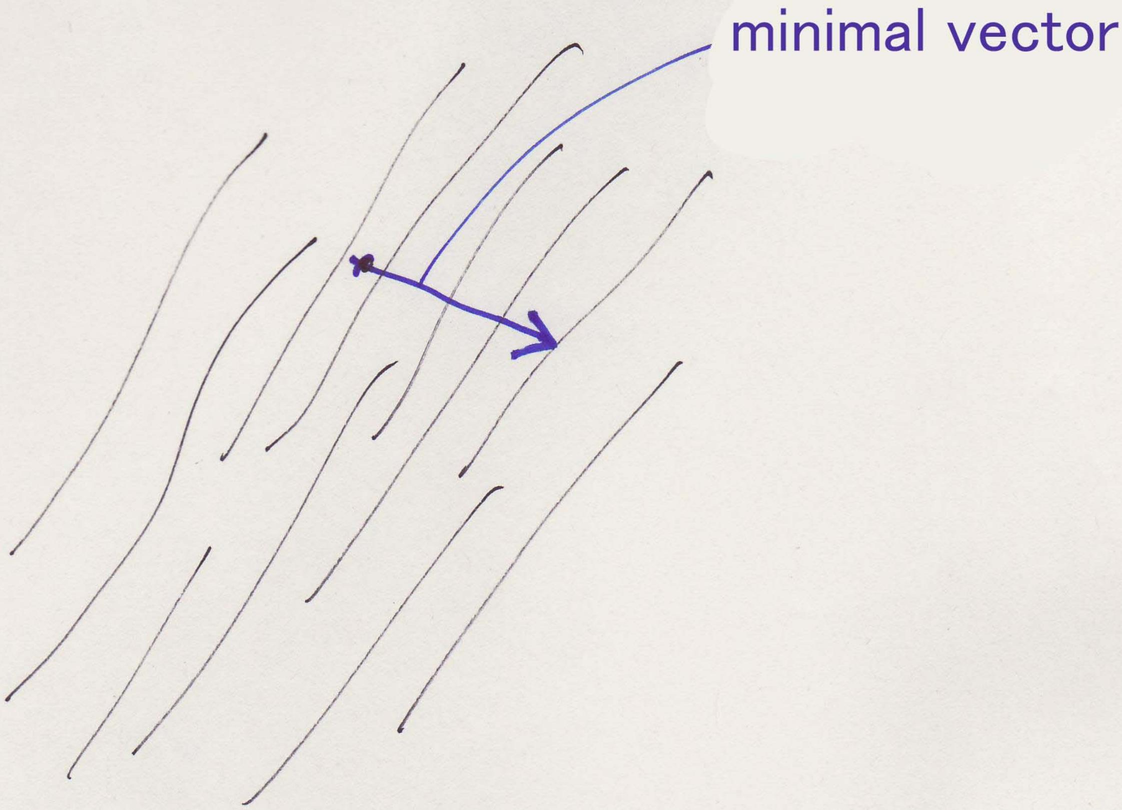


Suppose curves are distributed "continuously" at some density. (Curves may have a starting point and a terminal point.)

Let us call this distribution D symbolically.

* Consideration for "direction" is required, but here, let us not.

Please consider the following lecture as a "chat".



For each minimal vector,

let us call the correspondence between vector

and the “number” of curves in D that vector crosses

α.

$$D \xrightarrow{\text{"dual"}} \alpha$$

distribution of first dimensional figures | "dual" | first differential form

$\mathbb{R}^2$ inside

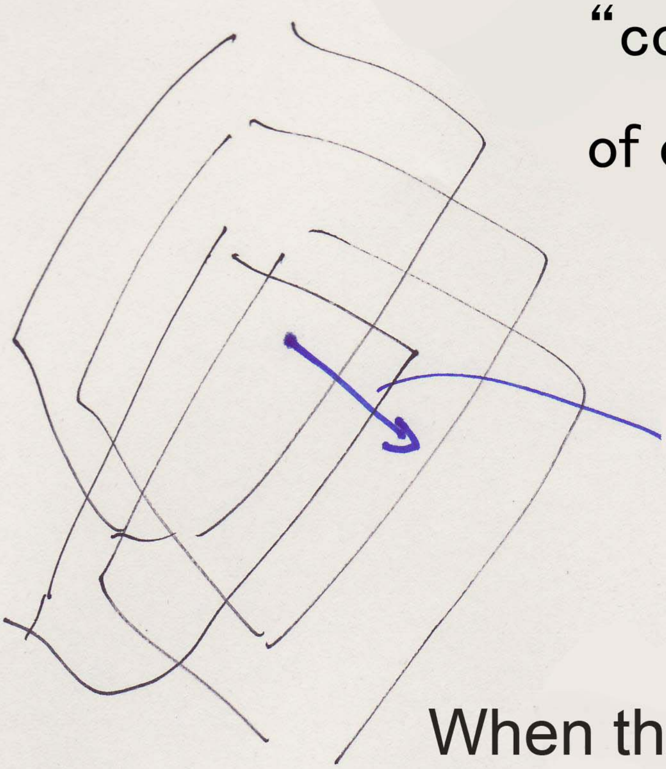
In a space, to give a first differential form …

$X = \mathbb{R}^3$

let us call

"continuous distribution"

of curves in D.



minimal vector

When the "number" of curves in D that minimal vector crosses is considered …

$$D \xrightarrow{\text{“dual”}} \alpha$$

distribution of two-dimensional figures inside $\mathbb{R}^3$

first differential form

k-order differential form

When
$$\begin{cases} \cdot \ \dim X = n \\ \cdot \ n = k + l \end{cases}$$

generally,

$$D \xrightarrow[\text{"dual"}]{\text{" "}} \alpha$$

distribution of l-dimensional figures

[k-dimensional minimal figure $\mapsto$ minimal number]

「k-dimensional differential form」

Cross-product

When

dim $X = n$

$D \xrightarrow{\text{dual}} \alpha$

distribution of $n-k$ dimensional figures

k -order differential form

$D' \xrightarrow{\text{dual}} \alpha'$

distribution of $n-k'$ dimensional figures

k'-order differential form

$k+k'$ dimensional differential form $\alpha \wedge \alpha'$

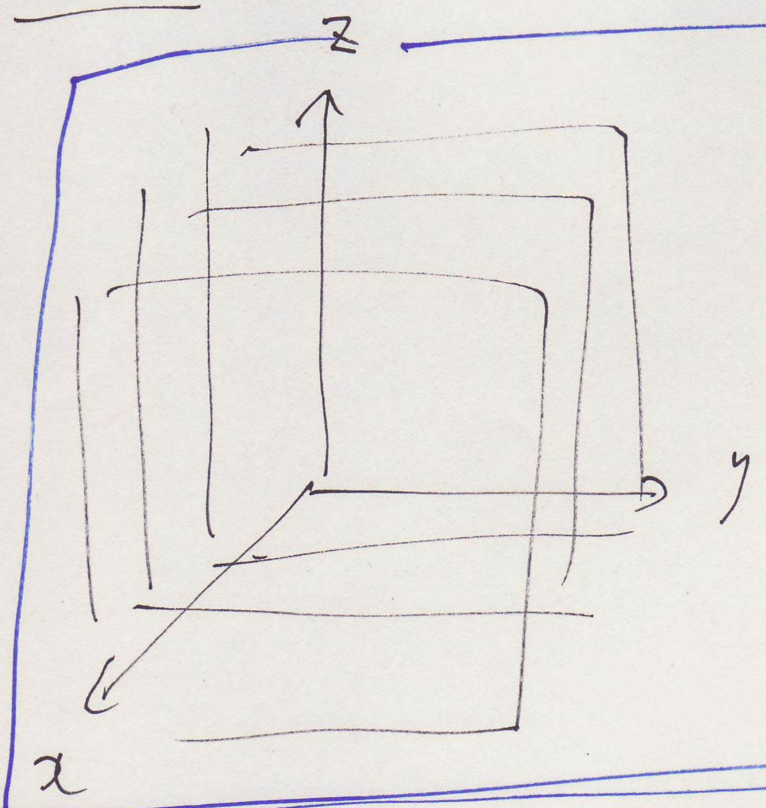
$D \cap D' \xrightarrow{\text{dual}} \alpha \wedge \alpha'$

distribution of $n-(k+k')$ dimensional figures

cross-product

can be defined as shown above.

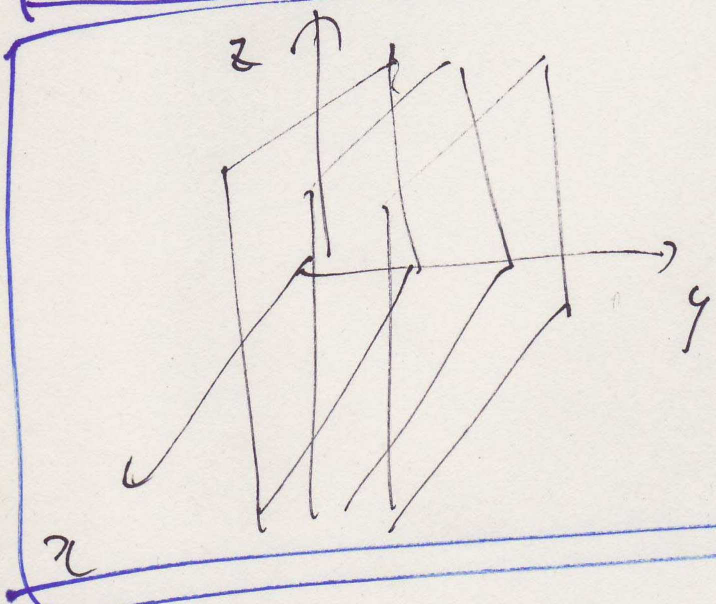
Ex.



dx first differential form

Corresponds to distribution of

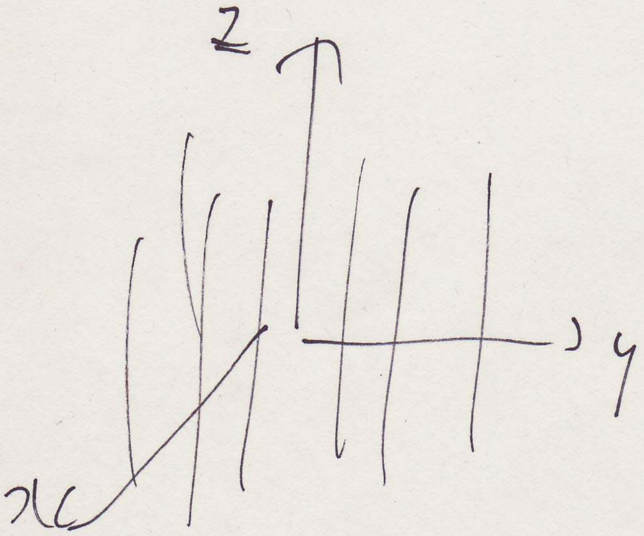
a plane $x = c$



dy first differential form

Corresponds to distribution of

a plane $y = c'$



$dx \wedge dy$ secondary differential form

Corresponds to distribution of

lines $\begin{cases} x = c \\ y = c' \end{cases}$

Exterior Differentiation

When $\dim X = n$

$D \overset{\text{dual}}{\rightsquigarrow} \alpha$

distribution of $n-k$ -dimensional figures

k -order differential form

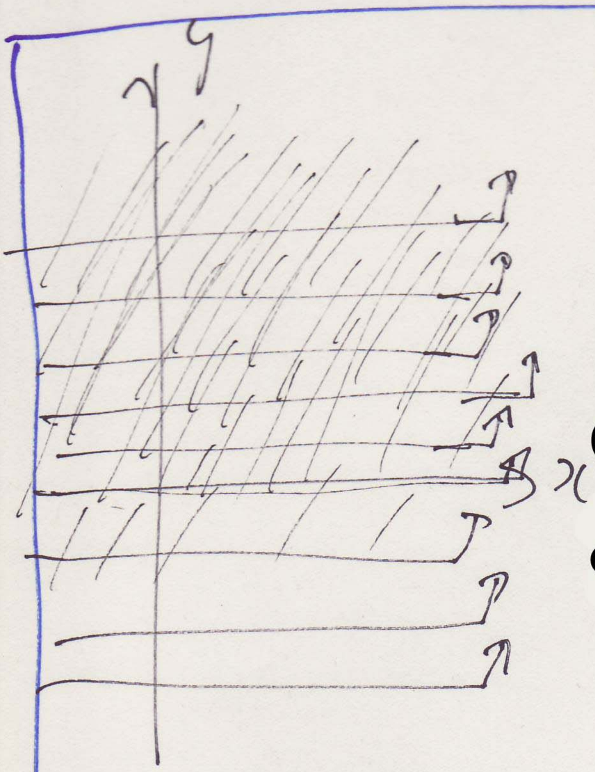
Then, $k+1$ -order differential form $d\alpha$

$\partial D \overset{\text{dual}}{\rightsquigarrow} d\alpha$

distribution of $n-k-1$ -dimensional figures

can be defined as shown above.

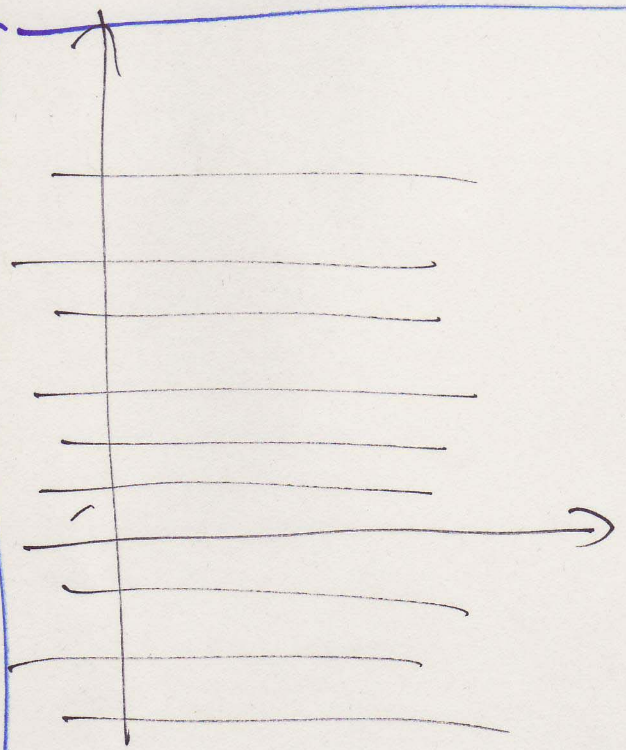
Example



y : function $\begin{pmatrix} a \\ b \end{pmatrix} \mapsto b$

0-order differential form

Corresponds to distribution of half-plane $y \geqq c$



dy first differential form

Corresponds to distribution of line $y = c$

Stokes' theorem

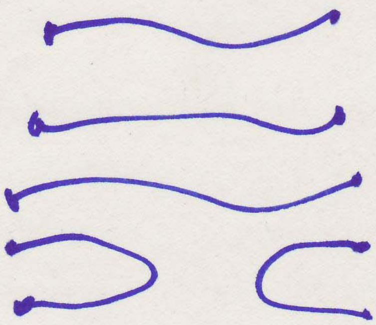
$$\int_{\gamma} d\alpha = \int_{\partial\gamma} \alpha$$

Suppose that

α corresponds to D

distribution

$$\underbrace{\partial(D\cap\gamma)}_{\text{connects both ends of first dimension}} = \underbrace{\partial D\cap\gamma}_{\text{left-hand side}} \cup \underbrace{D\cap\partial\gamma}_{\text{right-hand side}}$$



Leibniz's rule

$$d(\alpha \wedge \alpha') = d\alpha \wedge \alpha' \pm \alpha \wedge d\alpha'$$

$$D \rightsquigarrow \alpha$$

$$D' \rightsquigarrow \alpha'$$

$$\Longrightarrow D \cap D' \rightsquigarrow \alpha \wedge \alpha'$$

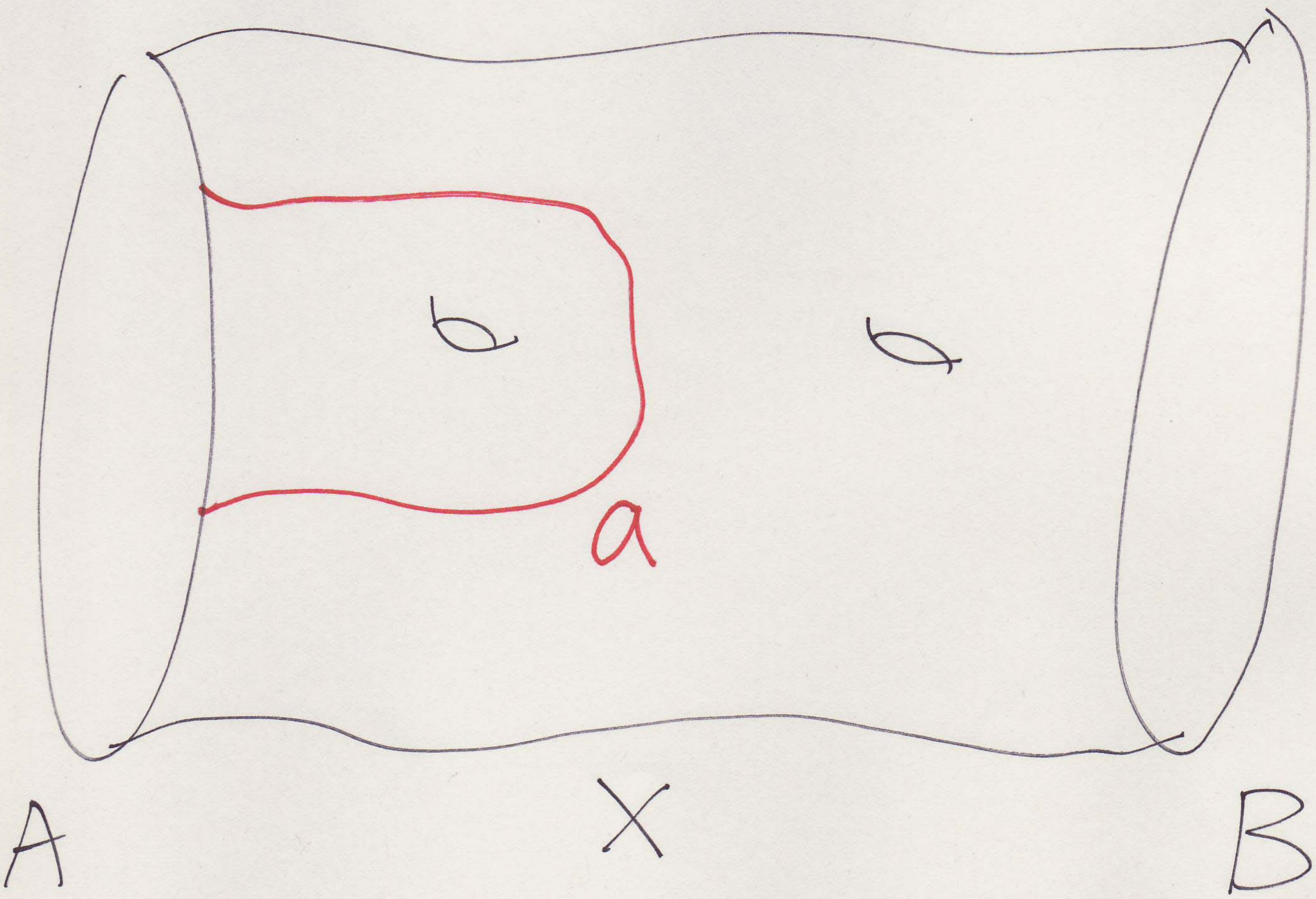
$$\partial(D \cap D') \rightsquigarrow d(\alpha \wedge \alpha')$$

$$\|$$

$$((\partial D) \cap D') \rightsquigarrow d\alpha \wedge \alpha'$$

$$\cup (D \cap (\partial D')) \rightsquigarrow \alpha \wedge d\alpha' \;//$$

Under these conditions ...



$\dim X = k + \ell$

$Z_k^{\mathbb{R}}(X, A)$ (weight $\mathbb{R}$, with direction) $\times \left\{ \alpha \,\middle|\, \begin{array}{l} k\text{-order differential form} \\ \circ\ d\alpha = 0 \\ \circ\ 0 \text{ near } A \end{array} \right\} \longrightarrow \mathbb{R}$

$a, \alpha \longmapsto \int_a \alpha$

Let's begin with Stokes' theorem ...

- $$H_k(X,A;\mathbb{R}) = Z_k^{\mathbb{R}}(X,A)/\sim$$

  $a \sim a' \Longleftrightarrow$ For a certain b ...

  $$\partial b = a - a'$$

- $$H^k_{DR}(X,A) = \left\{ \alpha \;\middle|\; \begin{array}{l} \cdot\ d\alpha = 0 \\ \cdot\ 0 \text{ near A} \end{array} \right\} \Big/ \left\{ d\beta \;\middle|\; \begin{array}{l} \beta : \\ 0 \text{ near A} \end{array} \right\}$$

  de Rahm cohomology

Under above conditions

"cognition of each other" "falls" into

$$H_k(X,A;\mathbb{R}) \times H^k_{DR}(X,A) \longrightarrow \mathbb{R}$$

de Rahm's theorem

theorem

$$H_k(X,A;\mathbb{R}) \times H^k_{DR}(X,A) \longrightarrow \mathbb{R}$$

is not degeneration.

Especially,

$$\dim H_k(X,A;\mathbb{R}) = \dim H^k_{DR}(X,A)$$

On the other hand, there is $\mathbb{R}$ version of Poincare duality.

$$\dim X = k + l$$

theorem (Poincare duality)

When

X has a direction,

$$H_k(X.A:\mathbb{R}) \times H_l(X.B:\mathbb{R}) \to \mathbb{R}$$

$$a.\quad x \longmapsto \text{"}\#(a \cap x)\text{"}$$

is not degenerated.

Especially,

$$\dim H_k(X,A:\mathbb{R}) = \dim H_l(X,B:\mathbb{R})$$

When compared...

$\left(\begin{array}{l} \dim X = k + l \\ X : \text{has a direction} \end{array} \right.$

$$H_l(X, B : \mathbb{R}) \xrightarrow{\cong} H^{k}_{DR}(X, A)$$

data about figures

data about differential form

This isomorphism is a function that corresponds to ...

$$D \xrightarrow{\text{dual}} \alpha$$

distribution of l -dimensional figures

k -order differential form

summary

◎ When duality is considered in geometry,

homology groups
cohomology groups
produce results.

◎ They work as a bridge between two worlds.

category: world of space X $\quad X \to Y$

factor $\Downarrow$

category: world of algebra $H_*(X)$ $\quad H_*(X) \to H_*(Y)$

They can work as a method to analyze geometry algebraically.

[an example of application:
fixed-point theorem]

On the other hand, homology
cohomology

have become methods used freely in algebra, analytics, and other disciplines of modern mathematics.

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