1

Geometrical Mind and Its Changes

Focusing on Topology and the Concept of Manifold

Mikio Furuta

First lecture

Geometry inside a plane or a space

↓

Geometry of curves, spaces, manifolds themselves

Second lecture

Measurement

duality

subject ← recognition tool

$\parallel$ measurement $\parallel$

recognition tool → subject

↓ one of the results in geometry

$H_*(X; \mathbb{Z}/2)$ $\quad H^*_{DR}(X)$

$H_*(X; \mathbb{R})$ $\quad (X$: manifold$)$

Poincaré duality : properties on chain

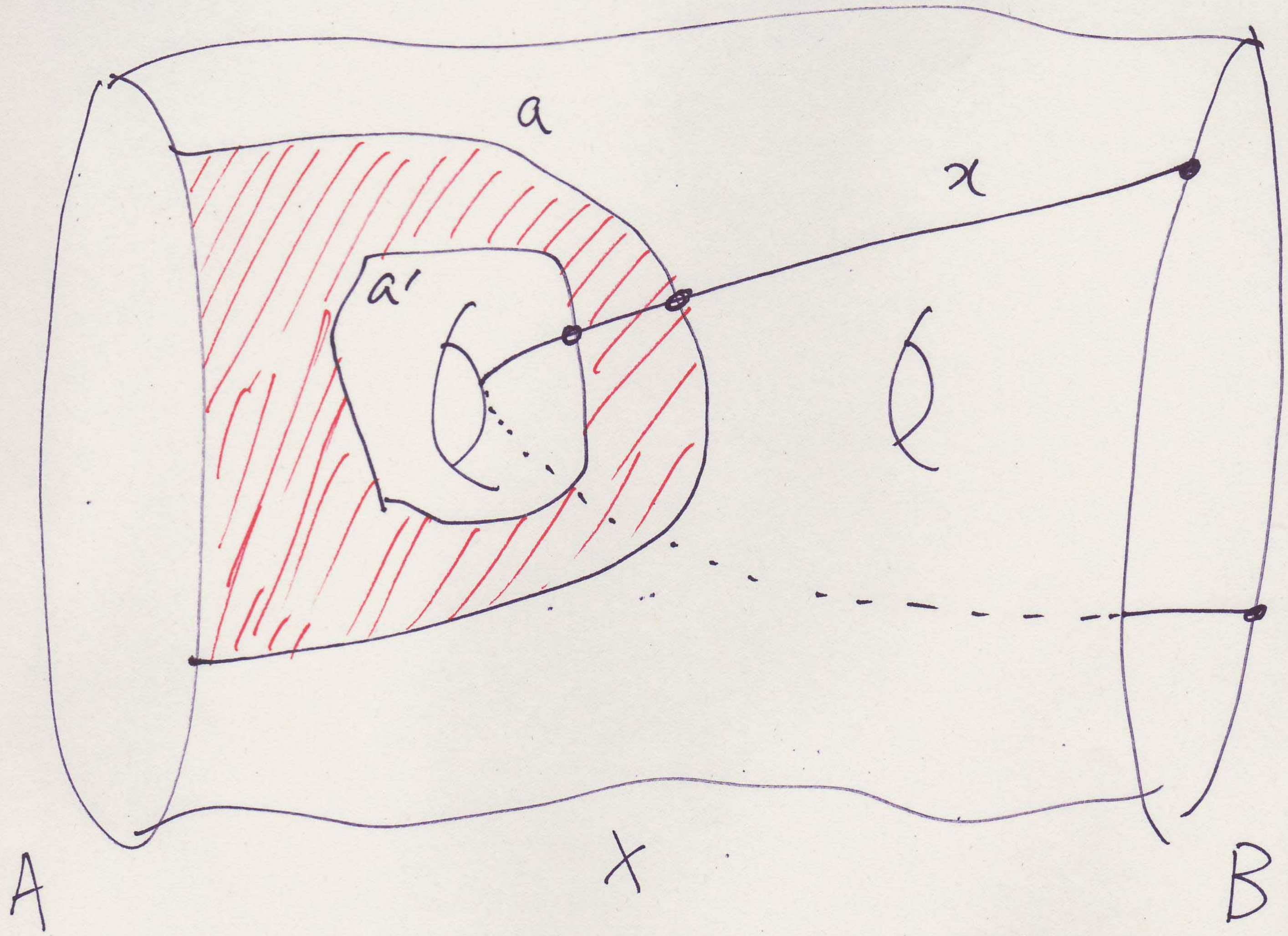
a & a' have same properties on number of intersections $\Rightarrow$ Between a & a', chain membrane can be made.

numerical, algebraic property

geometrical property

Review of the Previous Lecture

$\dim X = 2$



$$Z_1(X, A) \times Z_1(X, B) \longrightarrow \mathbb{Z}/2$$

$a \qquad x \qquad \#(a \cap x) \bmod 2$

When counting intersections, displace a little to be horizontal.

Form a membrane between a and a' $\Longleftrightarrow$ For any x $\#(a \cap x) \equiv \#(a' \cap x) \bmod 2$

Poincaré duality

Another Way of Expressing Poincaré Duality

$$H_1(X, A) \overset{\text{def}}{:=} Z_1(X, A) \big/ \sim$$

$H_1(X, A; \mathbb{Z}/2)$

$a \sim a' \overset{}{\underset{\text{def}}{\Longleftrightarrow}}$ A membrane can be formed.

$$\begin{array}{ccccc} H_1(X, A) & \times & H_1(X, B) & \longrightarrow & \mathbb{Z}/2 \\ [a] & & [x] & & \#(a \cap x) \bmod 2 \end{array}$$

This bilinear image is degenerate.

Upon the (finite body $\mathbb{Z}/2$)

Thus,

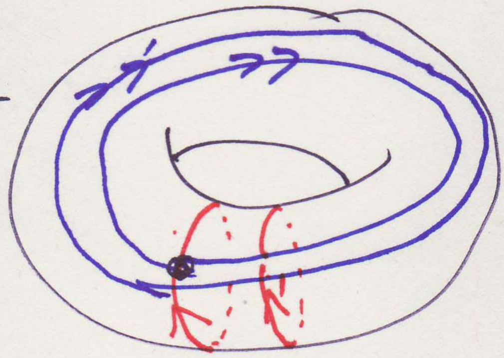
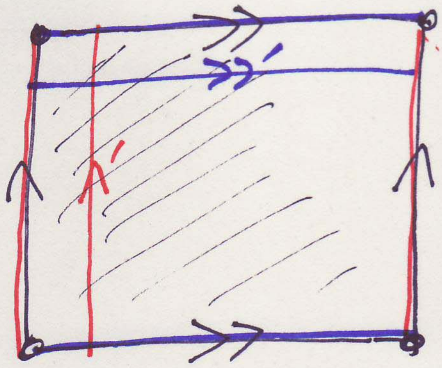
- $[a] = [a'] \Longleftrightarrow$ For all x $\#(a \cap x) \equiv \#(a' \cap x) \bmod 2$

and

- $[x] = [x'] \Longleftrightarrow$ For all a' $\#(a' \cap x) \equiv \#(a' \cap x') \bmod 2$

Thus, $H_1(X, A)$ and $H_1(X, B)$ recognize each other perfectly.

X = torus, in case $A = B = \phi$

$X = T^2$  = 

- $H_1(T^2; \mathbb{Z}/2)$ is $\mathbb{Z}/2$-contained two-dimensional vector space.
- Its origins are $[\uparrow'] = [\uparrow'']$, $[\twoheadrightarrow] = [\twoheadrightarrow']$

Number of intersections	$\uparrow'$ $\uparrow''$	$\twoheadrightarrow$ $\twoheadrightarrow'$
$\uparrow'$ $\uparrow''$	0 mod 2	1 mod 2
$\twoheadrightarrow$ $\twoheadrightarrow'$	1 mod 2	0 mod 2

Degeneration

Matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ mod 2 's determinant $\neq 0$ mod 2

Today's Lesson

The concept explained in the previous lecture :

- Poincaré duality

  [numerical/algebraic property $\Rightarrow$ geometrical consequence]

- $H_*(X)$ [X's properties are gathered in this vector space.]

Today's lecture is about how they developed in

"the Geometry of Curves, Spaces and Manifolds Themselves"

which was explained in the first lecture.

Examples of

1. How $H_*(X)$ can be used Algebraic topology

2. The birth of differential topology

— the discovery of exotic curves

3. How were they discovered?

Milner

4. $\partial($ 3D $) =$ 2D

5. Character figures

6. Strange properties of character figures

(7. Diversity of manifolds)

Cobordism theory

René Thom

8. In Conclusion

How $H_*(X)$ Is Used

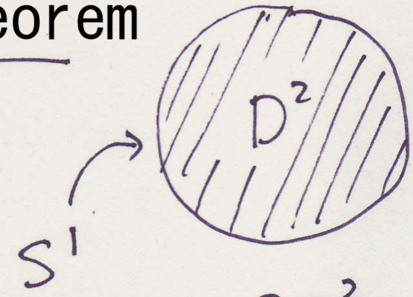
World of geometry $\longrightarrow$ World of algebra

$X \longmapsto H_*(X)$

$X \xrightarrow{f} Y \longmapsto H_*(X) \longrightarrow H_*(Y)$

Continuous mapping — Linear mapping

Theorem



$S^1 = \partial D^2$

There is no continuous mapping

that satisfies

$$f : D^2 \to S^1$$

$$\vdots$$

For any $p \in S^1 \ \rightarrow \ f(p) = p$

Proof

When using
$$\begin{cases} H_1(S^1; \mathbb{Z}/2) \cong \mathbb{Z}/2 \\ H_1(D^2; \mathbb{Z}/2) = \{0\} \end{cases}$$

$$S^1 \hookrightarrow D^2 \xrightarrow{f} S^1 \quad \xrightarrow{\text{“}H_1\text{”}} \quad \mathbb{Z}/2 \to \{0\} \to \mathbb{Z}/2$$

Identity mapping — Identity mapping

contradiction $\square$

The Birth of Differential Topology

Poincaré's question

When seeing algebraically,
A three-dimensional manifold cannot be distinguished from a three-dimensional sphere.
So are they not "equal"?

Poincaré conjecture subjected in 1904

If a three-dimensional manifold "lacks any boundary", "is compact" and "each loop in the space can be continuously tightened to a point", then it is "homeomorphic" to a three-dimensional sphere.

X and Y are homeomorphic $\Longleftrightarrow$ They are continuous mapping, $\begin{cases} X \xrightarrow{f} Y \\ Y \xrightarrow{g} X \end{cases}$ so they have inverse of each.

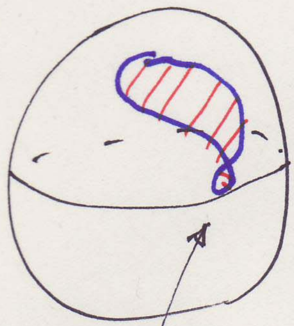
X is simply-connected $\Longleftrightarrow$ In a closed curve inside X, a circular membrane can be formed.

(A closed curve and a membrane can intersect with each other.)

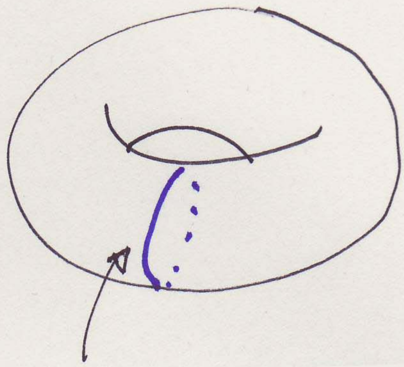
The History of Solving Poincaré Conjecture

- ## 2D Poincaré Conjecture

(Can be speculated from classification of spheres)



A circular disk membrane can be formed in any closed curve.



Whether it is a circular disk or not, any membrane cannot be formed even by a 2D chain.

A compact and borderless sphere with this property (=2D manifold) is only a (2D) sphere.

- Poincaré conjecture for dimensions greater than five — Smale 1960
- 4D Poincaré conjecture — Freedman 1981
- 3D Poincaré conjecture — Perelman 2002

Original Poincaré conjecture

Milnor's Exotic Sphere

(1956)

- Ordinary 7D sphere S^7 is a boundary for 8D circular disk D^8

$$\underbrace{S^7}_{\left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_8 \end{pmatrix} \middle| \sum x_i^2 = 1 \right\}} = \underbrace{\partial D^8}_{\left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_8 \end{pmatrix} \middle| \sum x_i^2 \leq 1 \right\}}$$

- Milnor put two parts of D^8's boundary together and made an 8D manifold with boundary, Z^8, whose $\Sigma^7 = \partial Z^8$ is just like an ordinary S^7.

How are S^7 and Σ^7 similar and different?

☆ S^7 and Σ^7 are homeomorphic !!

$$S^7 \underset{g}{\overset{f}{\rightleftarrows}} \Sigma^7$$

f and g are continuous mapping and each has its own inverse mapping, $f \cdot g$

☆ S^7 and Σ^7 are not diffeomorphic !!

$$S^7 \underset{g}{\overset{f}{\rightleftarrows}} \Sigma^7$$

f and g are smooth mapping and they do not have their inverse mapping $f \cdot g$.

(Partial differetiation is possible if displayed in coordi- nates.)

How Can It Be Proved?

☆ S^7 and Σ^7 are homeomorphic.

$\because$) Specifically, make $S^7 \underset{g}{\overset{f}{\rightleftarrows}} \Sigma^7$.

f and g are continuous and an inverse of each other.

(In fact, f and g are smooth at all the points except one. The only point that isn't cannot be smooth anyhow.)

☆ S^7 and Σ^7 are not diffeomorphic.

D^8  Z^8

S^7 Σ^7

$\Downarrow$

If they are diffeomorphic, they can be stuck smoothly.

$X^8 :=$ 

Study this property of X and draw a contradiction.

8D Manifold X^8 Has Some Contradictions

Hirzebruch signature theorem

(for 8D, borderless, compact and directional manifold X)

$$\underbrace{\sigma(X)}_{\text{signature}} = \frac{1}{45}\left(\underbrace{7\,p_2(X) - p_1^2(X)}_{\text{character figure}}\right)$$

In case of Milnor's example

$$\begin{cases} \cdot\ \sigma(X) = 1 \\ \cdot\ p_1^2(X) = 36 \\ \cdot\ p_2(X) : \text{integer} \end{cases} \Rightarrow 7 \mid 81$$

$\Longrightarrow$ contradiction

$\square$

What Is a Signature? $\sigma(x)$

X^8 : It is compact, and has border and direction.

$$\boxed{H_4(X;\mathbb{R}) \times H_4(X.\mathbb{R}) \longrightarrow \mathbb{R}}$$

$H_4(x:\mathbb{R})$ is in $\mathbb{R}$, a vector space of finite dimension.

For a base arbitrarily chosen,

$[a_1], [a_2], \cdots, [a_m]$

$a_i \quad a_j \longmapsto \#(a_i \cap a_j)$

Matrix
$$\begin{pmatrix} \#(a_1 \cap a_1) & \cdots & \#(a_m \cap a_1) \\ \vdots & & \vdots \\ \#(a_1 \cap a_m) & & \#(a_m \cap a_m) \end{pmatrix}$$

Symmetric matrix and holomorphic (Poincaré duality)

$\sigma(x) :=$ number of positive proper value $-$ number of negative proper value

Summary of Further Lectures

Milnor discoverd that "homeomorphic" and "diffeomorphic" are different concepts.

⇒ The birth of new geometry that considers diffeomorphic manifolds to be the "same".

Differential topology

In this view,

Number of manifolds which are homeomorphic to n-sphere

n	1	2	3	4	5	6	7	8	9	10	11	12	13
Number	1	1	1	?	1	1	28	2	8	6	992	1	3

↑ Milnor's first discovery

Lesson From Here

Further points

Milnor used the formula by Hirzebruch shown below.

$\sigma(X)$ = "expression by character figure"

signature

$H_*(X)$'s structure determines this number

(X is 8D.)

☆ What is a character figure?

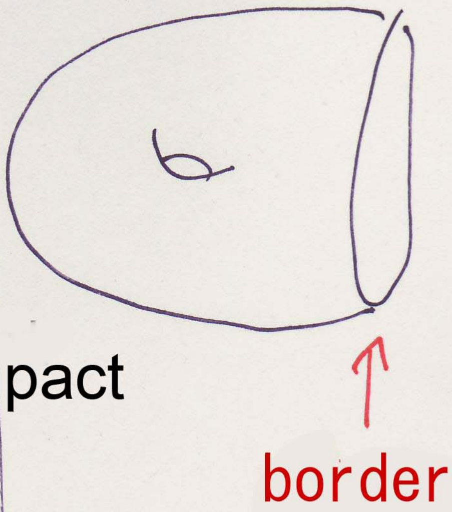
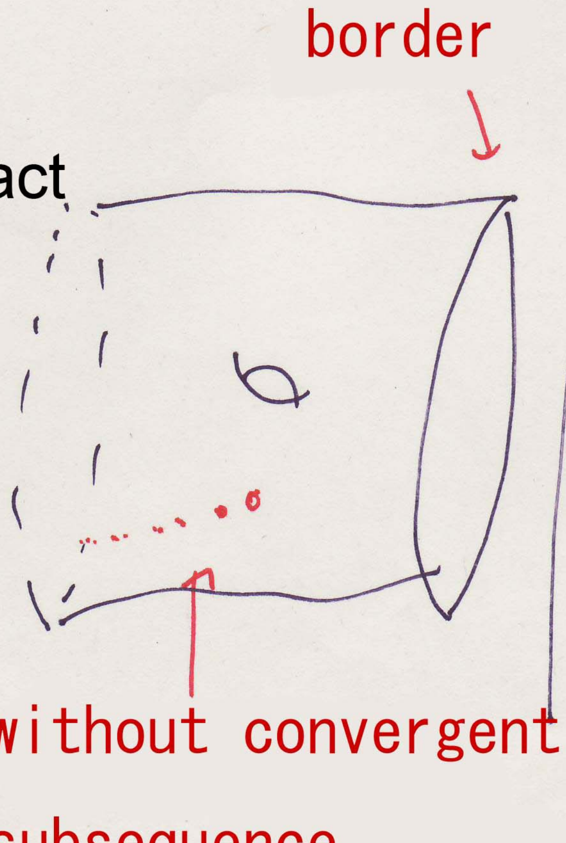
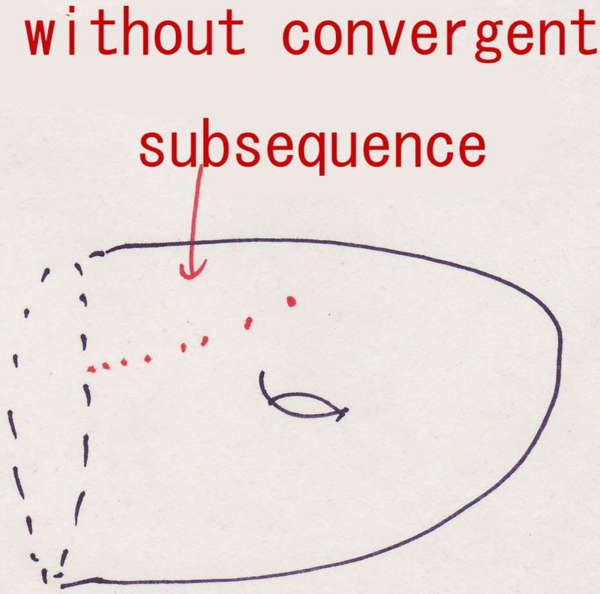
☆ How did Hirzebruch prove the formula above?

Let's explain this

not by $\boxed{\text{8D}}$ but by $\boxed{\text{2D.}}$

with direction $\boxed{H_*(X; \mathbb{R})}$ Direction is not assumed. $\boxed{H_*(X; \mathbb{Z}/2)}$

Closed Curve = Compact and Borderless Two-dimensional Manifold

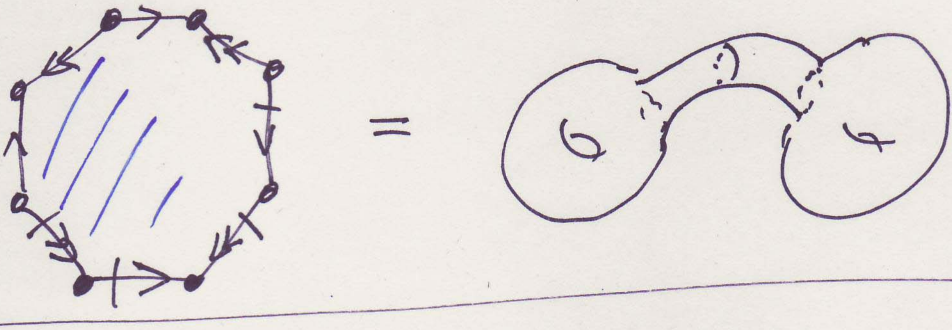
	With border	Without border
Compact		
Non-Compact		

Examples of a Closed Curve

1

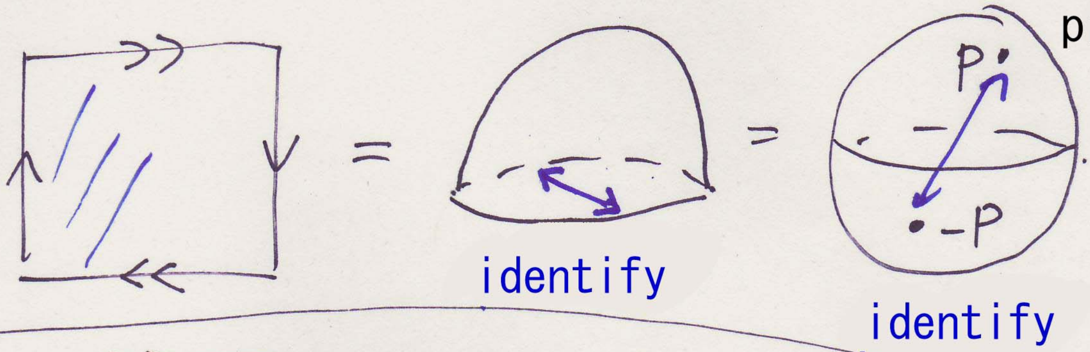
= torus T^2

2



= $T^2 \# T^2$

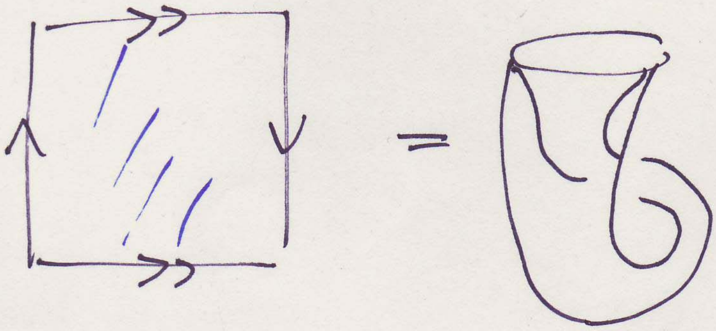
3



projective plane

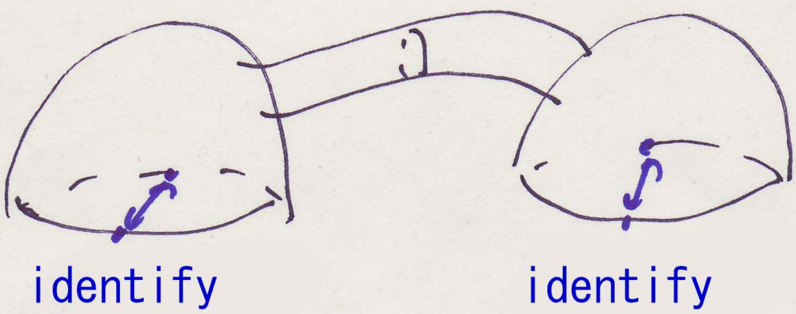
$\mathbb{R}P^2$

4



= Klein bottle

5

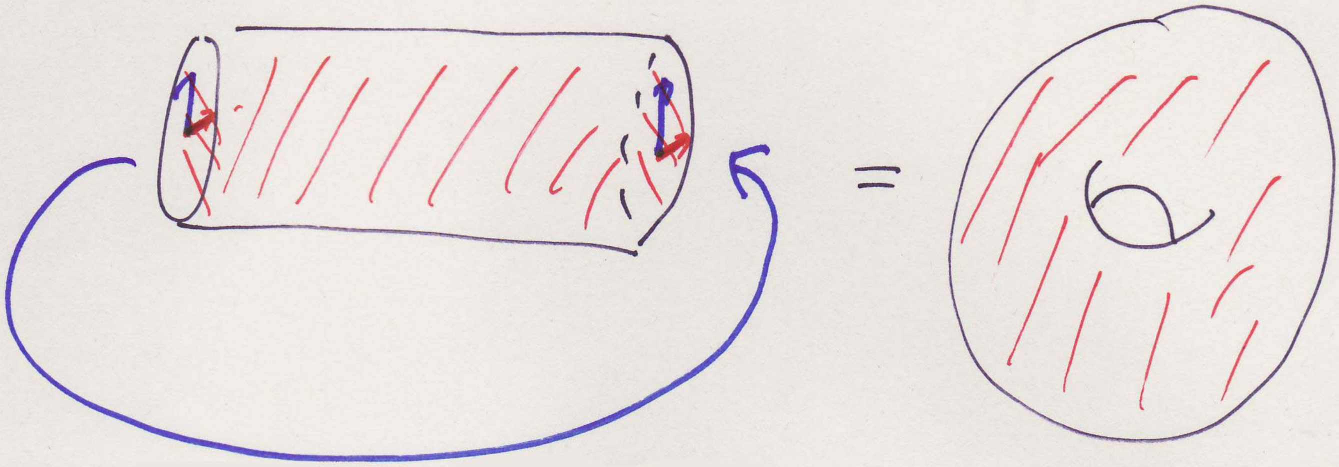


$\mathbb{R}P^2 \# \mathbb{R}P^2$

4 = 5 (Both are two Möbius bands attached together.)

Examples of 3D Compact Manifolds with Border

I

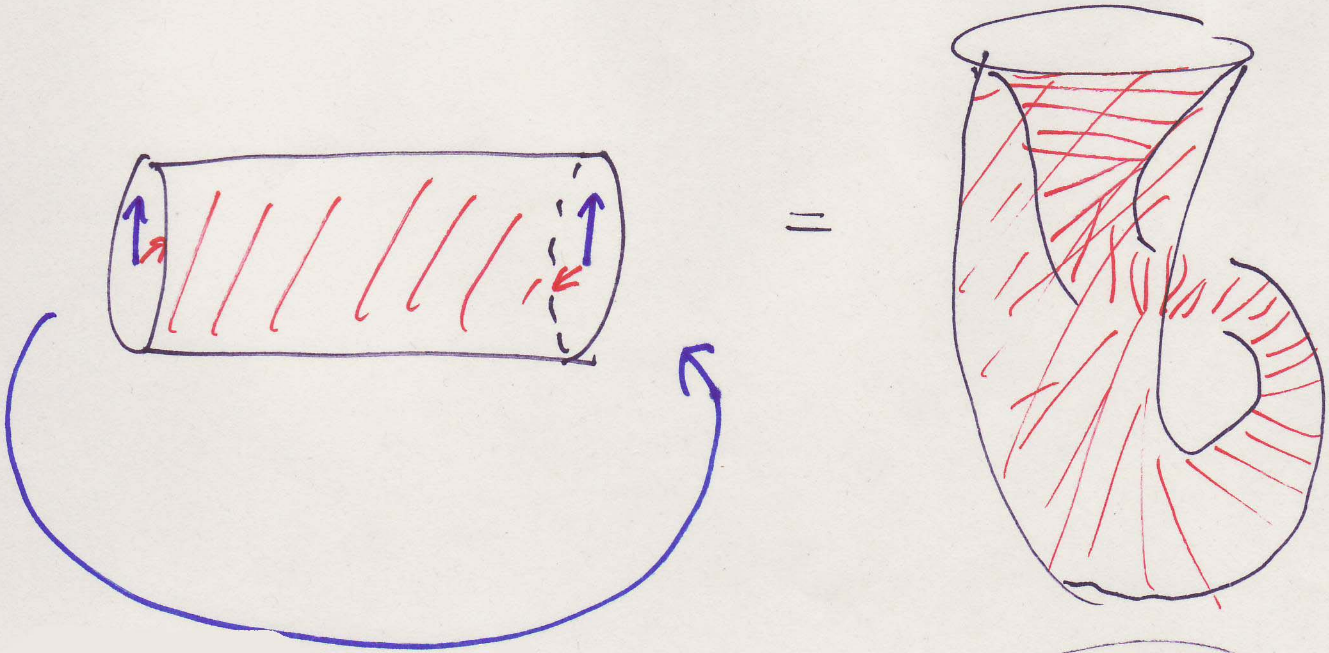


Turn around and identify.

solid torus

The border is a torus.

II



Turn around, get "a mirror image" and identify.

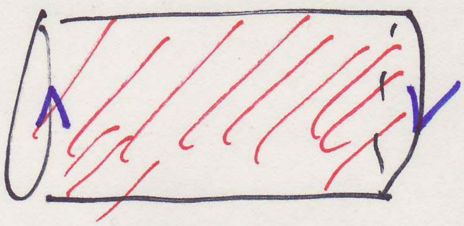
The border is a Klein bottle.

"Examples of Closed Curves" Which Can Be Drawn as a Border of 3D Compact Manifold

① ∂  $=$  T^2

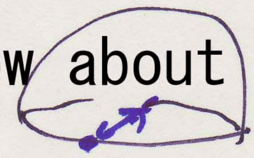
③ ∂  $=$ 

$T^2 \# T^2$

④ $=$ ⑤ ∂  $=$ 

Klein bottle

$\mathbb{R}P^2 \# \mathbb{R}P^2$

Q. In ③, how about 

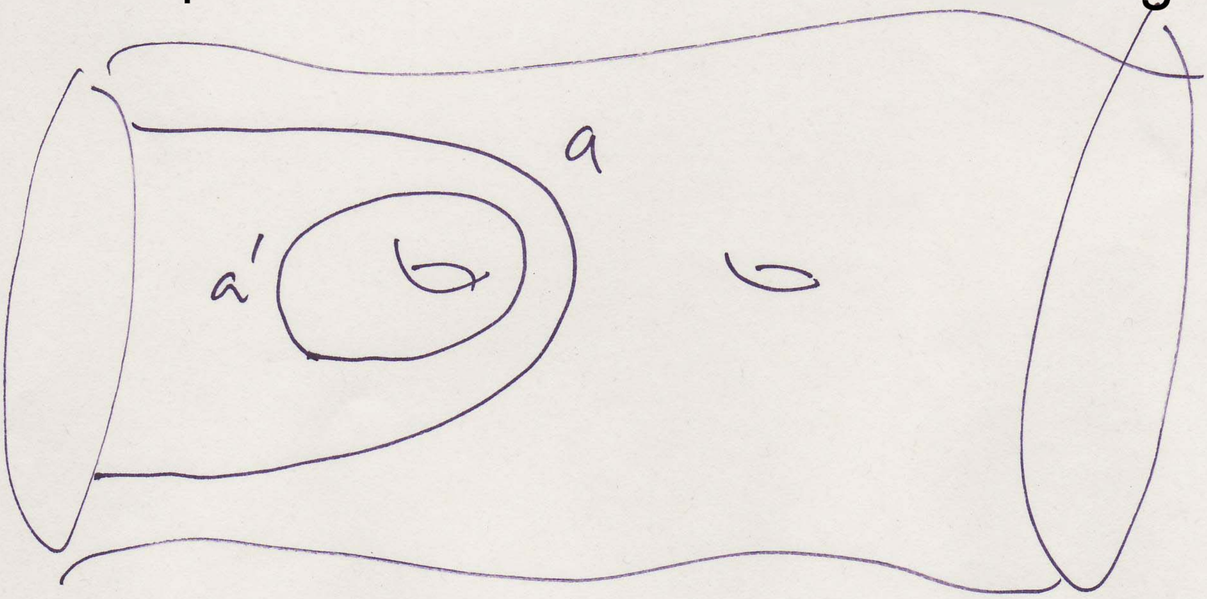
$\mathbb{R}P^2$?

--- Speaking From the Conclusion,

A. Impossible!!

$\mathbb{R}P^2$ cannot be drawn as border.

The question and the answer above is similar to the question and the answer following.



Between a and a', can a membrane be formed?

||

Is $a \cup a'$ a boundary of a membrane?

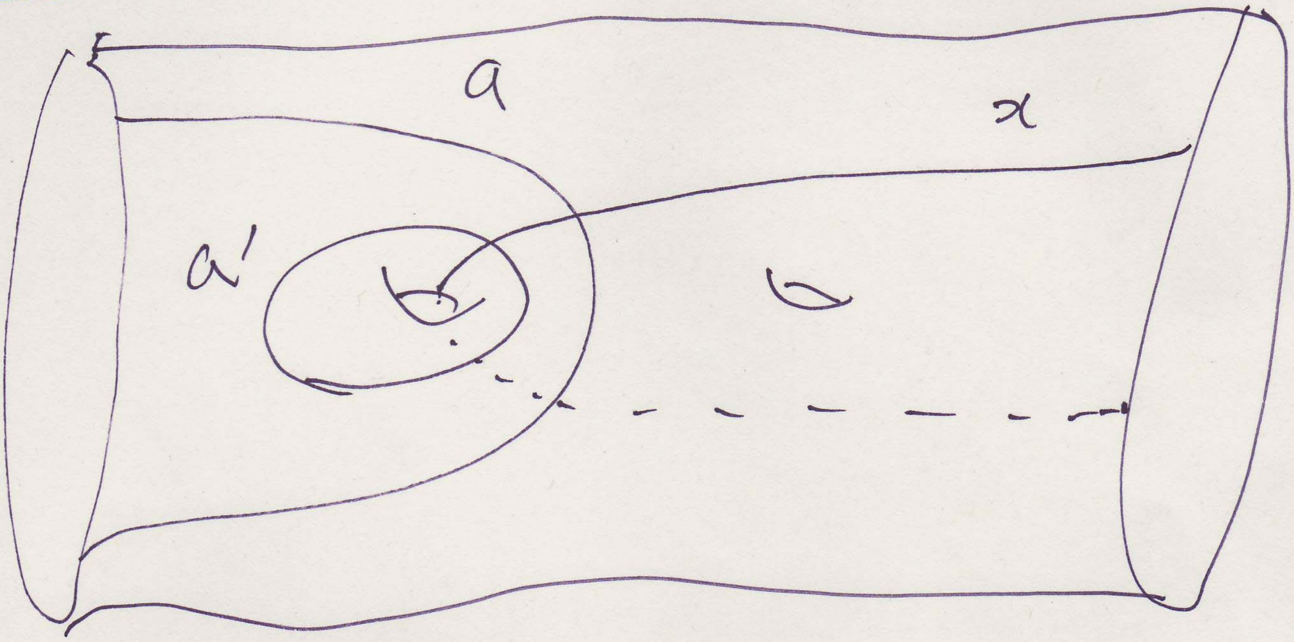
A. 1

(required condition)

If, for any x

$$\#(a \cap x) \not\equiv \#(a' \cap x) \mod 2$$

impossible



A. 2

(sufficient condition)

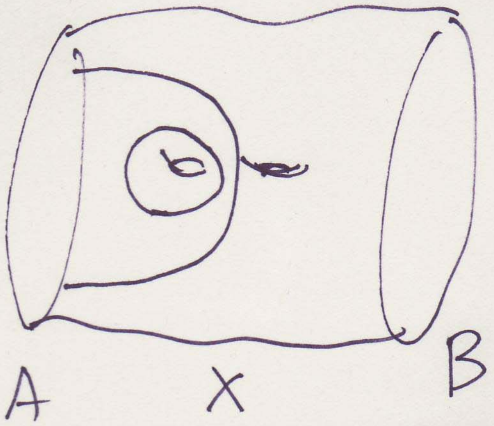
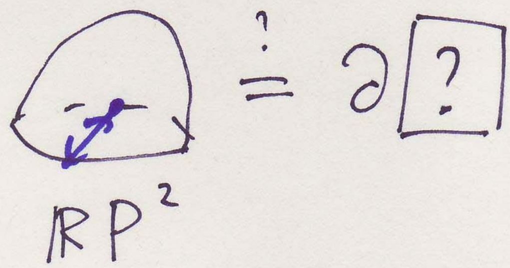
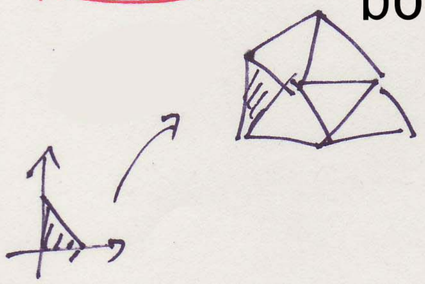
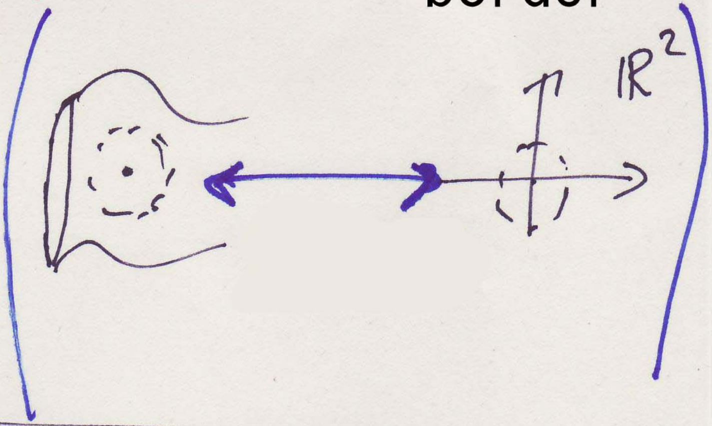
If, for any x

$$\#(a \cap x) \equiv \#(a' \cap x) \mod 2$$

possible

[Poincaré duality]

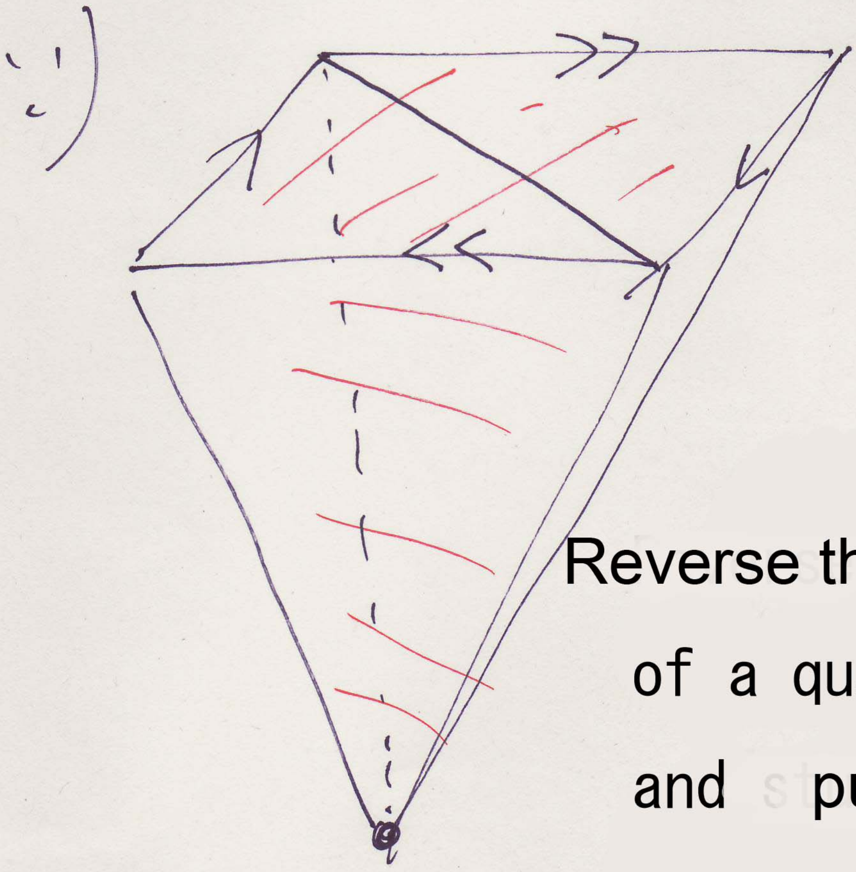
Differences Between This Question and the Previous Question

	Previous time	This time
		
Sub-ject	chain and its border 	manifold and its border 
World	Think of subjects as being inside a single manifold X In other words, X is the world.	Consider all the possibilities for every subject .

In fact, $\mathbb{RP}^2$ can be a border of

"3D compact chain".

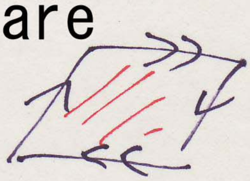
We can say that this is a more precise problem.



Reverse the "facing sides" of a quadrangular pyramid, and put them together.

$\Rightarrow$ The figure then consists of two triangular pyramids.

Their borders are

 $= \mathbb{RP}^2$

Not manifold!!

$\square$

In “this problem”, there are

- required conditions to draw as a border
- sufficient conditions to draw as a border

and if the fact that both of these conditions match is proved, then the problem can be completely solved.

Description of required conditions is given below.
The fact that they are also sufficient conditions is only to be mentioned.

This also happened in the description of

- Poincaré duality.
- de Rahm theory

What To Think Instead of Intersections: Character Figure

Direction of proving that

$\mathbb{R}P^2$ cannot ∂Z
be drawn
in form of

$\underline{\text{step 1}}$ To the closed curve X

apply the character figure $w_1^2(X) \in \mathbb{Z}/2\mathbb{Z}$

$\underline{\text{step 2}}$ If $\boxed{X = \partial Z}$

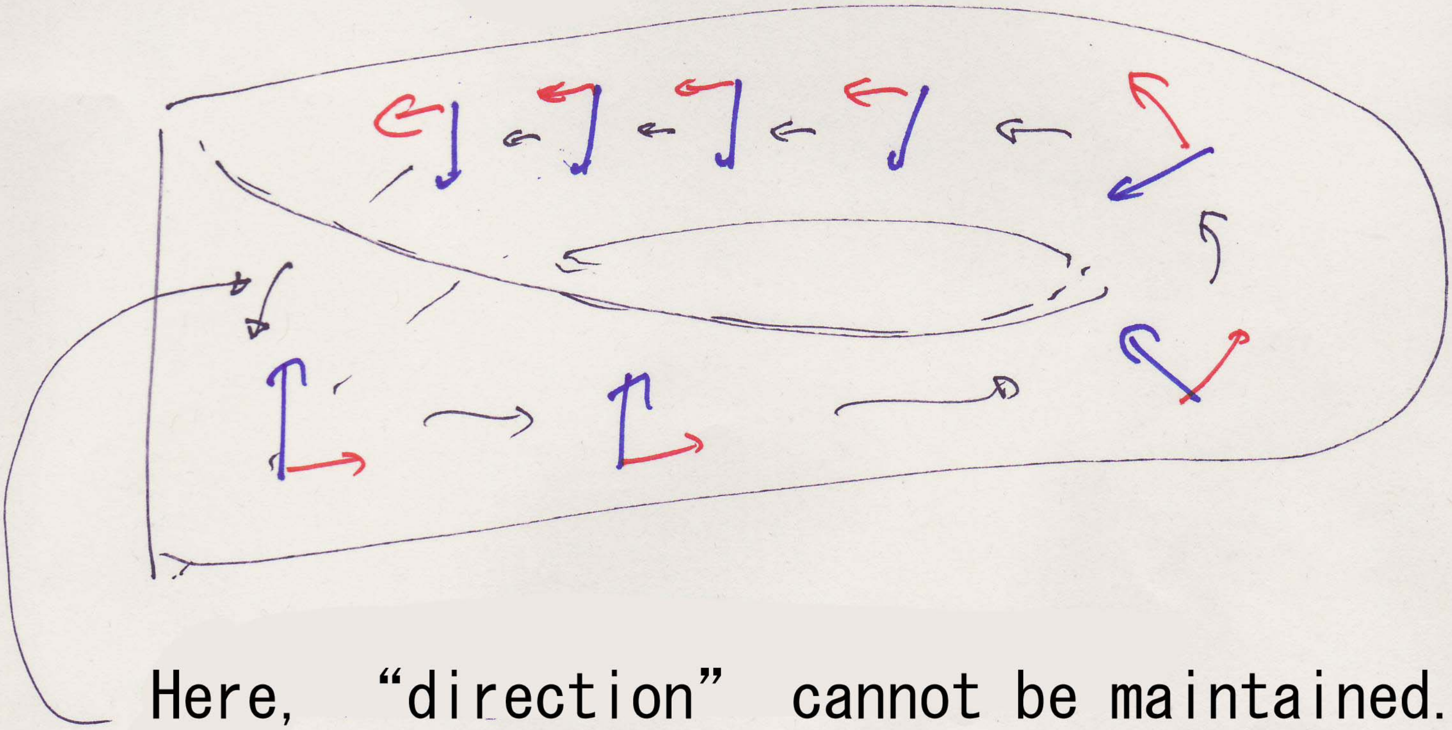
prove $\underline{w_1^2(X) = 0 \in \mathbb{Z}/2\mathbb{Z}}$

$\underline{\text{step 3}}$ Check $\underline{w_1^2(\mathbb{R}P^2) = 1 \in \mathbb{Z}/2\mathbb{Z}}$

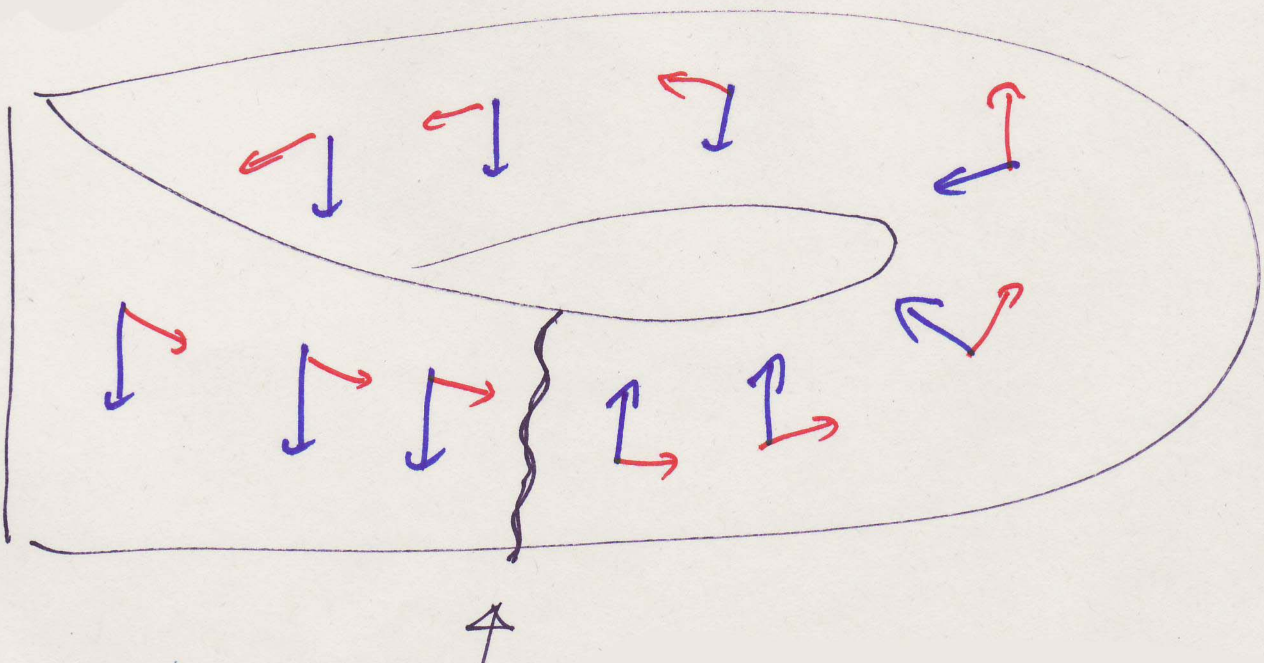
$\square$

What is the disturbing induction of direction?

In case of Möbius band,

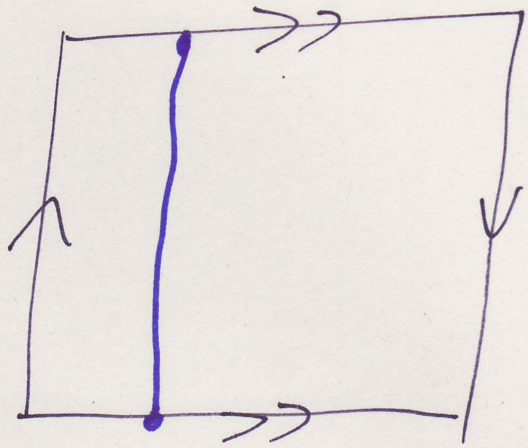


Here, “direction” cannot be maintained.



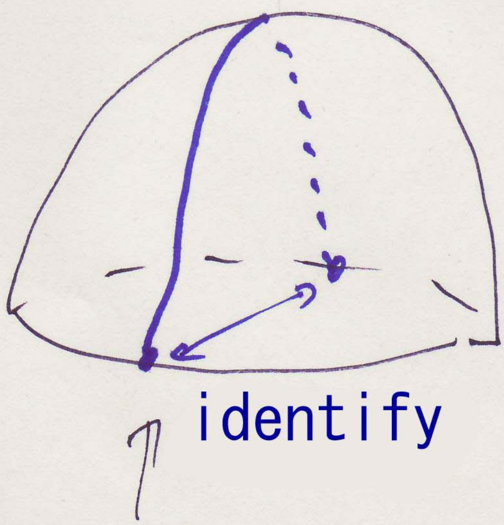
If you cut here, direction would be determined.

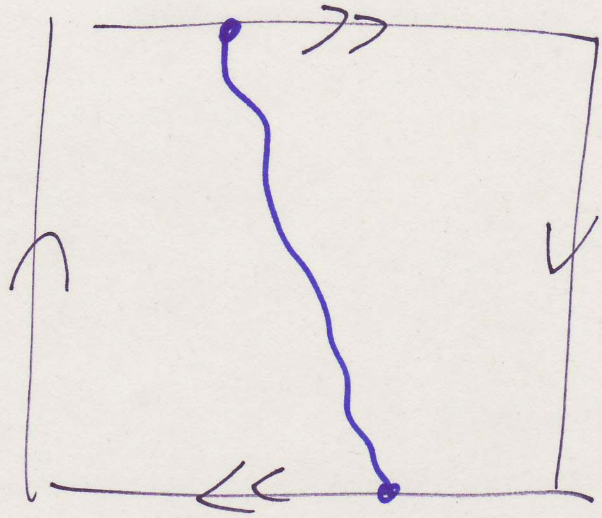
In case of a Klein bottle



cut here

In case of a projective plane





identify

cut here

$w_1^2(X) \in \mathbb{Z}/2$'s Definition

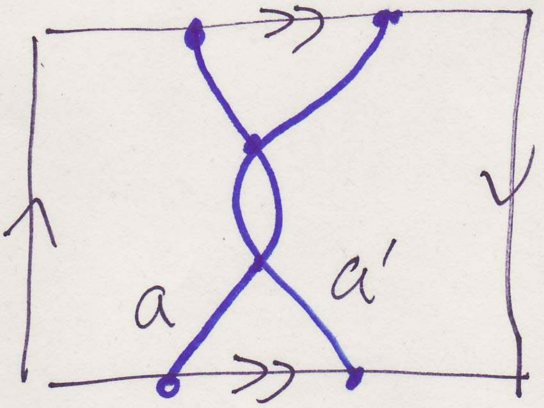
Cut in a way that both sides of cut surfaces are in opposite direction.

- Cut at two sections so each cut surface has a direction, and name them a, a'

- The number of intersecting points of those two cut sections. mod 2, is $w_1^2(X)$

$$\boxed{w_1^2(X) = \#(a \cap a') \bmod 2}$$

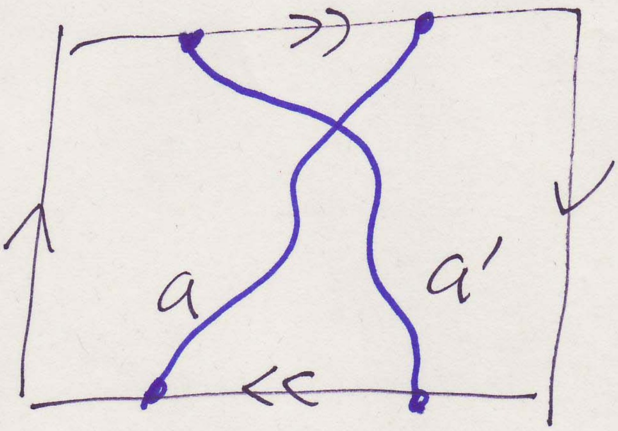
Klein bottle



$\#(a \cap a') = 2$

$w_1^2(\text{Klein bottle}) \equiv 0 \in \mathbb{Z}/2\mathbb{Z}$

Projective plane



$\#(a \cap a') = 1$

$w_1^2(\mathbb{R}P^2) \equiv 1 \in \mathbb{Z}/2\mathbb{Z}$

$w_1^2(X)$'s Properties

(1) First, $\#(a \cap a') \bmod 2 \in \mathbb{Z}/2$ is determined regardless of how a, a' are taken.

Therefore, it depends only on X.

Hence, the symbol $w_1^2(X)$ is justified.

---

(2) When $X = \partial Z$

$$\#(a \cap a') \bmod 2 = 0$$

Therefore, $w_1^2(X) = 0 \in \mathbb{Z}/2$

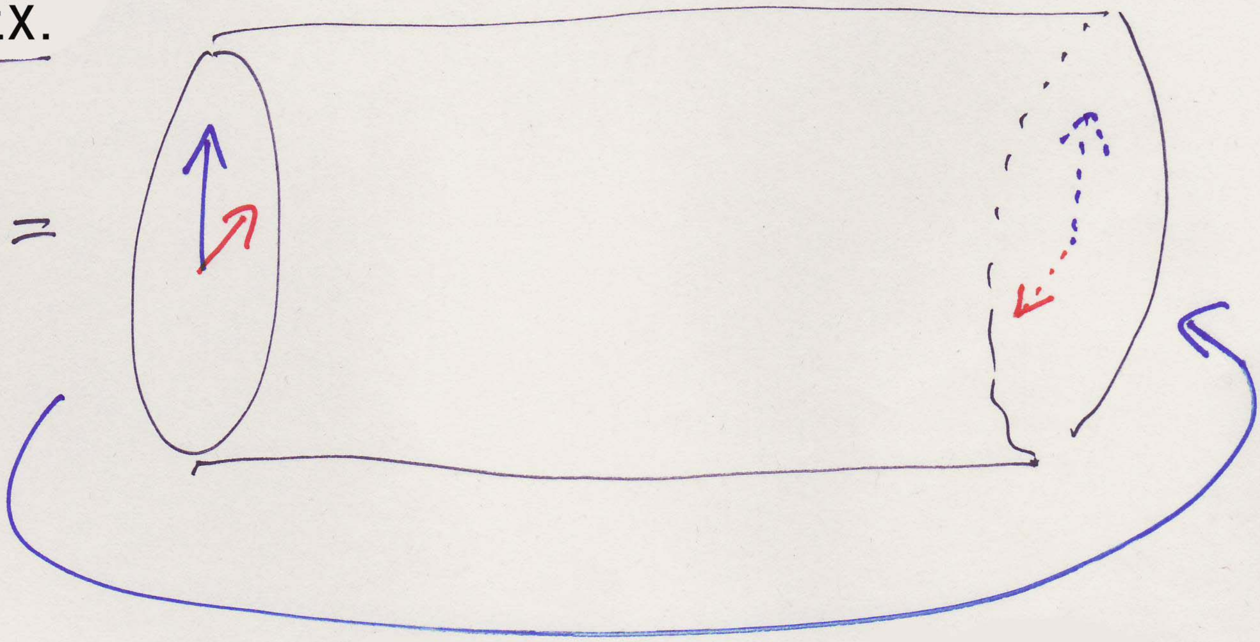
In fact, proofs of (1) and (2) are similar.

Here, only (2) is to be explained.

When $X = \partial Z$

Ex.

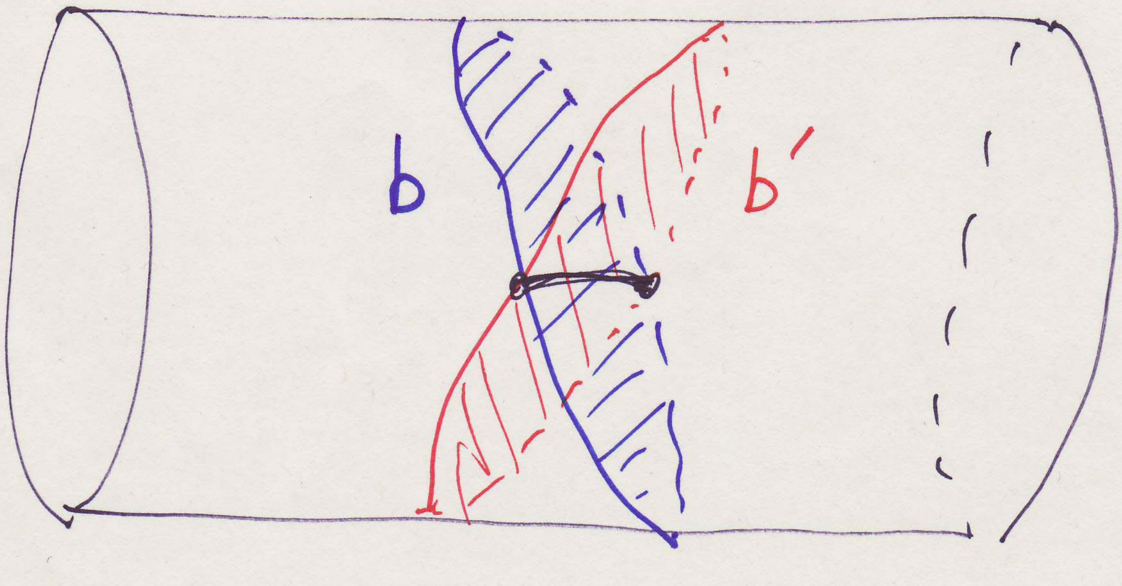
$Z =$



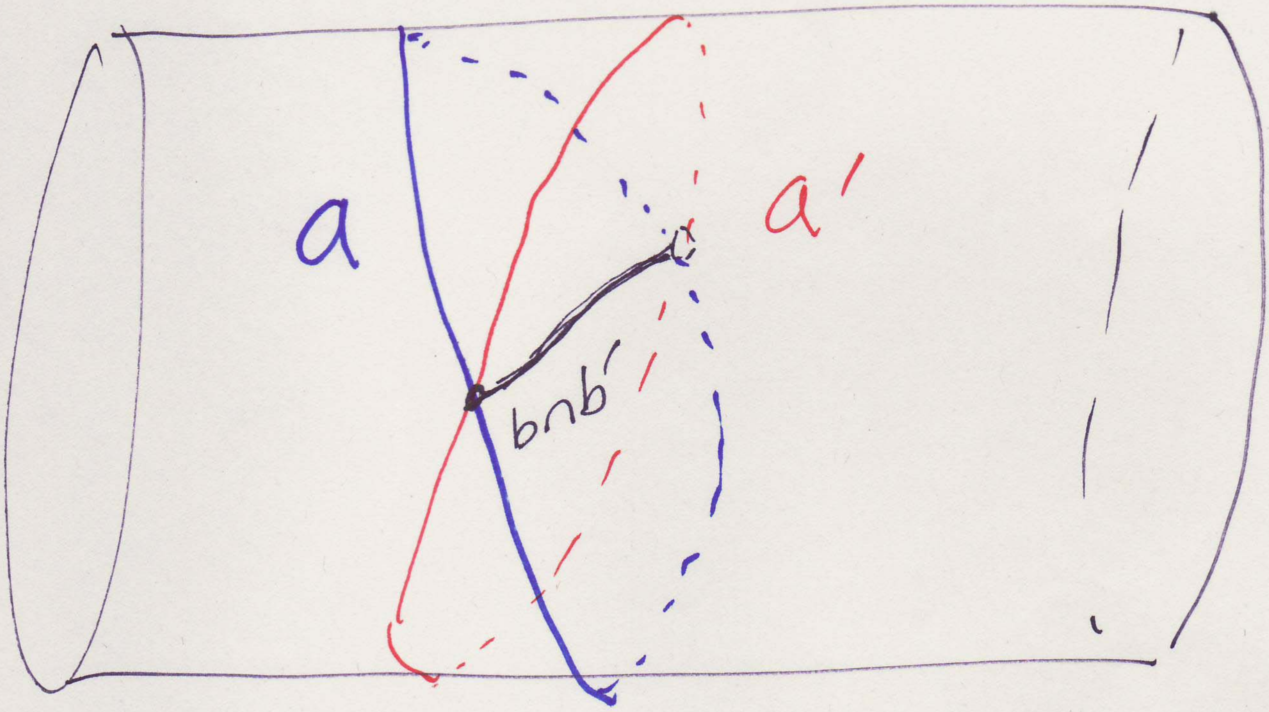
Take a mirror image and " put them together"

Cut Z at two sections so the cut surfaces have directions and then name them

b, b'



$a = \partial b$ $\quad a' = \partial b'$



$$\partial(\underbrace{b \cap b'}) = a \cap a'$$

First dimension, so the number of boundaries is even.

Q.E.D

$\square$

Now, Essential Conditions for Being a Boundary are Understood!

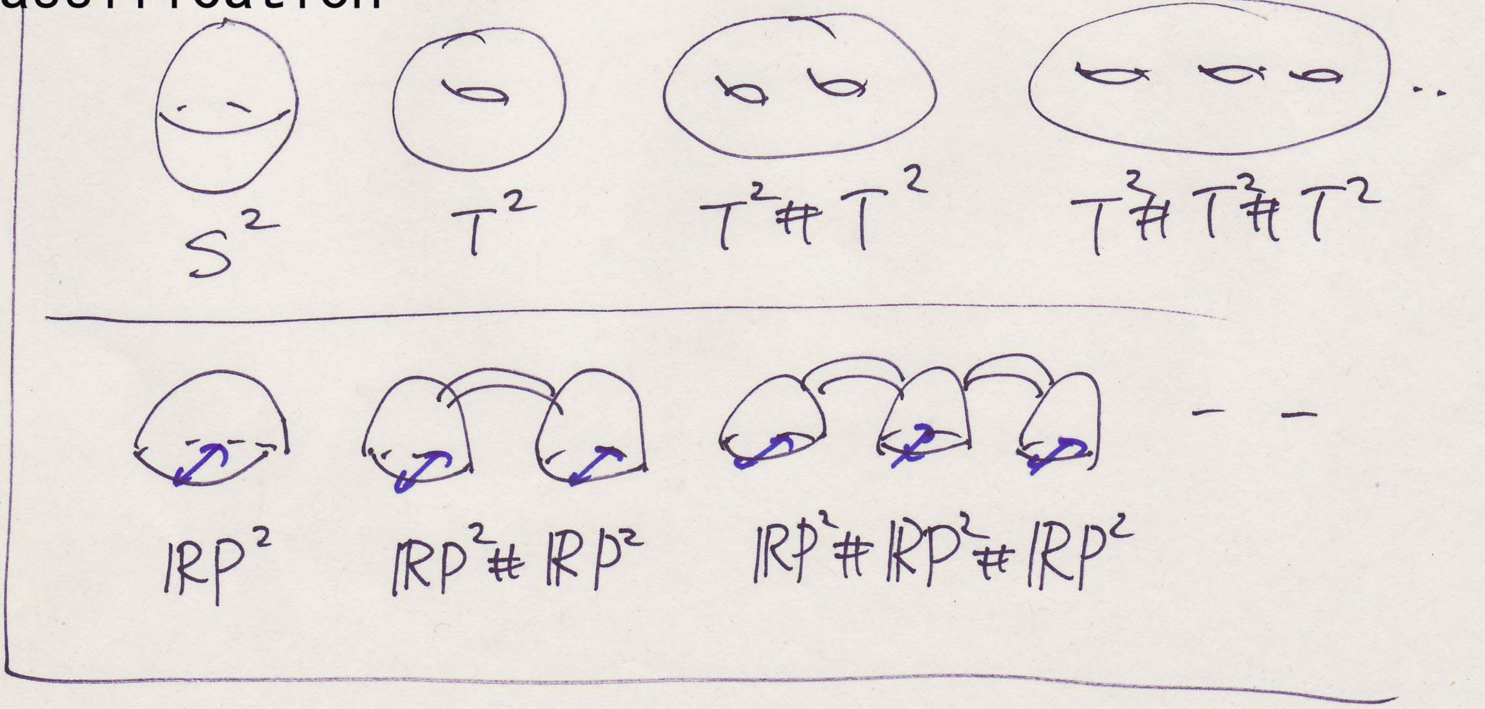
Then, would conditions for NOT being a boundary be proved?

(A more profound, geometrical question!)

Answer: Yes

By using the classification theorem of closed spheres, it is easily proved to be "yes".

Classification



- See the diagram of classification.

- $w_1^2(\underbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{\text{odd number}}) = 1 \in \mathbb{Z}/2$

So, $\underbrace{\mathbb{R}P^2 \# \cdots \# \mathbb{R}P^2}_{\text{even number}}$ cannot be written in the form of ∂Z.

- For other closed spheres X except above,

$X = \partial Z$-becoming Z can be practically made.

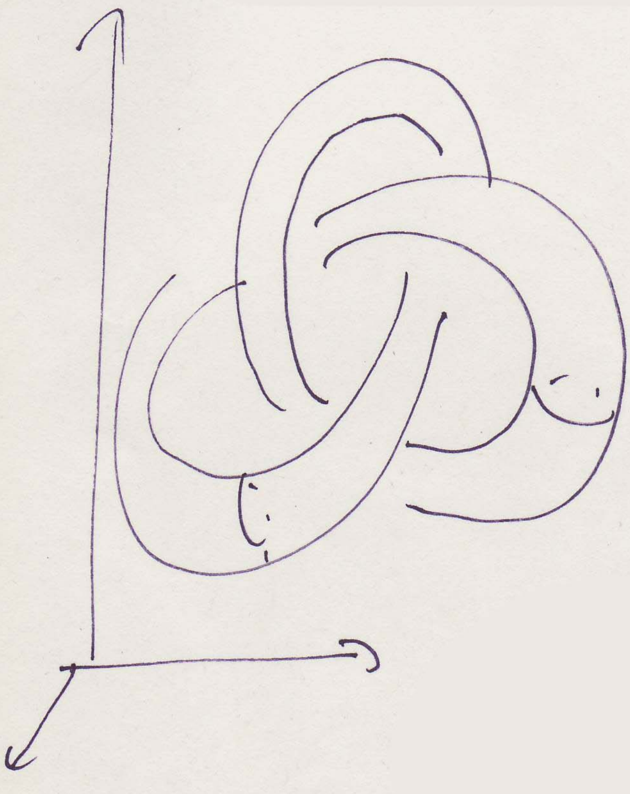
As above, for a closed sphere X,

$\boxed{\text{Theorem}}$ $w_1^2(X) = 0 \iff X = \partial Z$-becoming Z exists.

How About in Higher Dimensions?

Enumerating manifolds is practically impossible.

Ex. A three-dimensional manifold is , at least, as diverse as knots.



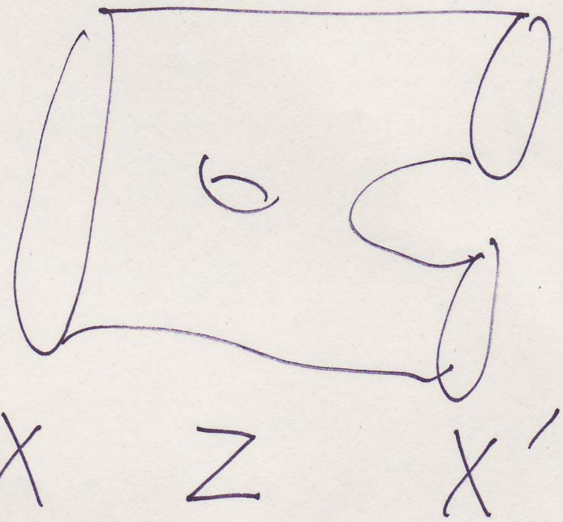
Cut out an area of a knot from $\mathbb{R}^3$, and put it back in a different way.

Then, a new three-dimensional manifold is formed.

Reversing the Idea: Thom's Cobordism Theorem

- Classify manifolds roughly.

- $X \sim X' \iff$ For a certain Z,



can be drawn.

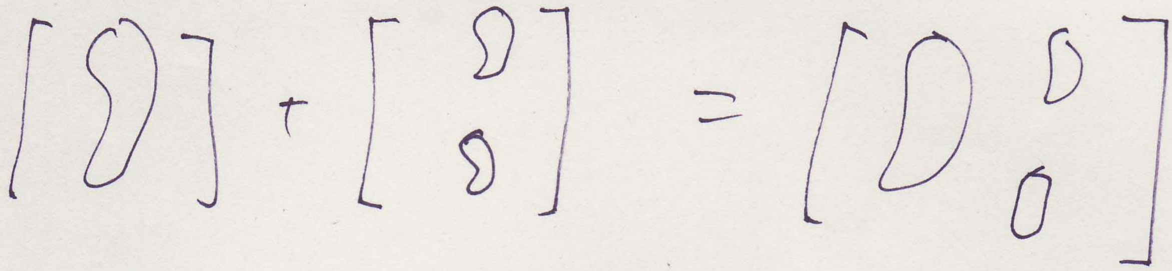
$$\partial Z = X \cup X'$$

- Represent equivalent class by $[X]$

- $\Omega_n := \{\text{n-dimensional closed manifold}\} / \sim$

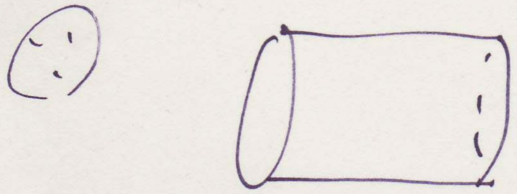
- Addition

$$[X] + [X'] = [X \cup X']$$



Ex.

$$[X] + [X] = [\phi] \quad \text{i.e} \quad 2[X] = 0$$



$$\partial(X \times [0,1]) = X \times \{0\} \cup X \times \{1\}$$

$\Omega_n = \{[X]\}$ is a vector space on the finite body, $\mathbb{Z}/2\mathbb{Z}$.

Note. $\bigoplus_{n=0}^{\infty} \Omega_n$ is a ring. $[X]\cdot[Z] = [X \times Z]$

What Was Understood as a Result of Profound Speculations by Thom, Milnor, and Others

$$\Omega_n \times \left\{ \begin{array}{l} \text{"all" the} \\ \text{possibilities} \\ \text{of character} \\ \text{figures } (*) \end{array} \right\} \longrightarrow \mathbb{Z}/2$$

$$[X], \quad w_1^2 \longmapsto w_1^2(X)$$

The bilinear mapping above is defined, and it is degenerate.

$$\Omega_0 \cong \mathbb{Z}/2 \quad \Omega_1 \cong \{0\} \quad \Omega_2 \cong \mathbb{Z}/2 \quad \Omega_3 \cong \{0\}$$

$$\cdots$$

(*) Precisely, not "all" but adequate parts are needed.

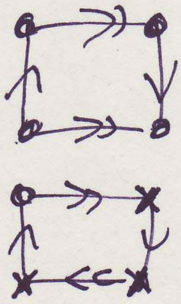
Back in the Case of Two-dimensions...

$w_1^2(X)$'s Strange Properties

	S^2	T^2	$T^2 \# T^2$	$\mathbb{R}P^2$	Klein bottle $\mathbb{R}P^2 \# \mathbb{R}P^2$
$w_1^2(X)$	$0 \bmod 2$	$0 \bmod 2$	$0 \bmod 2$	$1 \bmod 2$	$0 \bmod 2$
$\dim_{\mathbb{Z}/2} H_1(X;\mathbb{Z}/2)$	0	2	4	1	2
(Euler number)	2	0	-2	1	0
$\dim H_1$ or Euler number $\bmod 2$	$0 \bmod 2$	$0 \bmod 2$	$0 \bmod 2$	$1 \bmod 2$	$0 \bmod 2$

(In polygonal decomposition,

Euler number = $\boxed{\#\text{ of points}}$ − $\boxed{\#\text{ of sides}}$ + $\boxed{\#\text{ of surfaces}}$)



In fact, when X : a closed sphere

Theorem

Euler number $\equiv w_1^2(X) \bmod 2$

An amount related to $H_*(X, \mathbb{Z}/2)$

character figure

This is a theorem parallel to the Hirzebruch signature theorem.

Signature $=$ Expression by a character figure

$$\sigma(X^8) = \frac{1}{45}(7p_2(X^8) - p_1^2(X^8))$$

Ω_*-using Proof

- In case of a closed sphere (2D), classification is known, so the theorem can be easily verified independently.
- Also, a direct proof is possible.
- However, as shown below, Ω_2 structure using proof is also possible.

step1

$$\Omega_2 \longrightarrow \mathbb{Z}/2$$

$$[X] \longmapsto [X\text{'s Euler number}] \bmod 2$$

is semi-isomorphic. ($\mathbb{Z}/2$ linear mapping)

step2

$$\Omega_2 \longrightarrow \mathbb{Z}/2$$

$$[X] \longmapsto w_1^2(X)$$

is also semi-isomorphic ($\mathbb{Z}/2$ linear mapping)

step3 Both matches on Ω_2 's base.

Proof of Signature Theorem by Hirzebruch

For a directed, closed manifold,

- form a similar structure of Ω_n and name it $\Omega_n^{\mathbb{Z}}$

- $$\bigoplus_{n=0}^{\infty} \Omega_n \longrightarrow \mathbb{Z}$$
  $$[X] \longmapsto \sigma(X)$$
  is semi-isomorphic (as a ring)

- Form a semi-isomorphism made by a good coordination of "character figures"
  $$\bigoplus_{n=0}^{\infty} \Omega_n \longrightarrow \mathbb{R}$$
  $$[X] \longmapsto L(X)$$

- By using Thom' s cobordism theorem, check if $\bigoplus_{n=0}^{\infty} \Omega_n$ matches its origin as a ring.

Summary

Series of discussions by Poincaré → resulted as a vector space $H_*(X:\mathbb{Z}/2)$

homology class

⇓

This $H_*(X:\mathbb{Z}/2)$ can be used as a tool.

Ex.
- $S^1 \hookrightarrow D^2 \xrightarrow{f} S^1$ does not exist. (id)
- fixed-point theorem (This can be easily induced from above.)

Question of "cobordism" → resulted as a vector space Ω_*

cobordism class

⇓

This Ω_* can be used as a tool.

ex. signature theorem for each manifolds

Geometry ○○○○.

To look over the whole while focusing on connections betweenthe parts

What is “a part” and what is “the whole”?

It changes by time and circumstances. Sometimes, an ultra-global consideration that cannot be imagined from the starting point is made.

The END