Mathmetics of Phenomena

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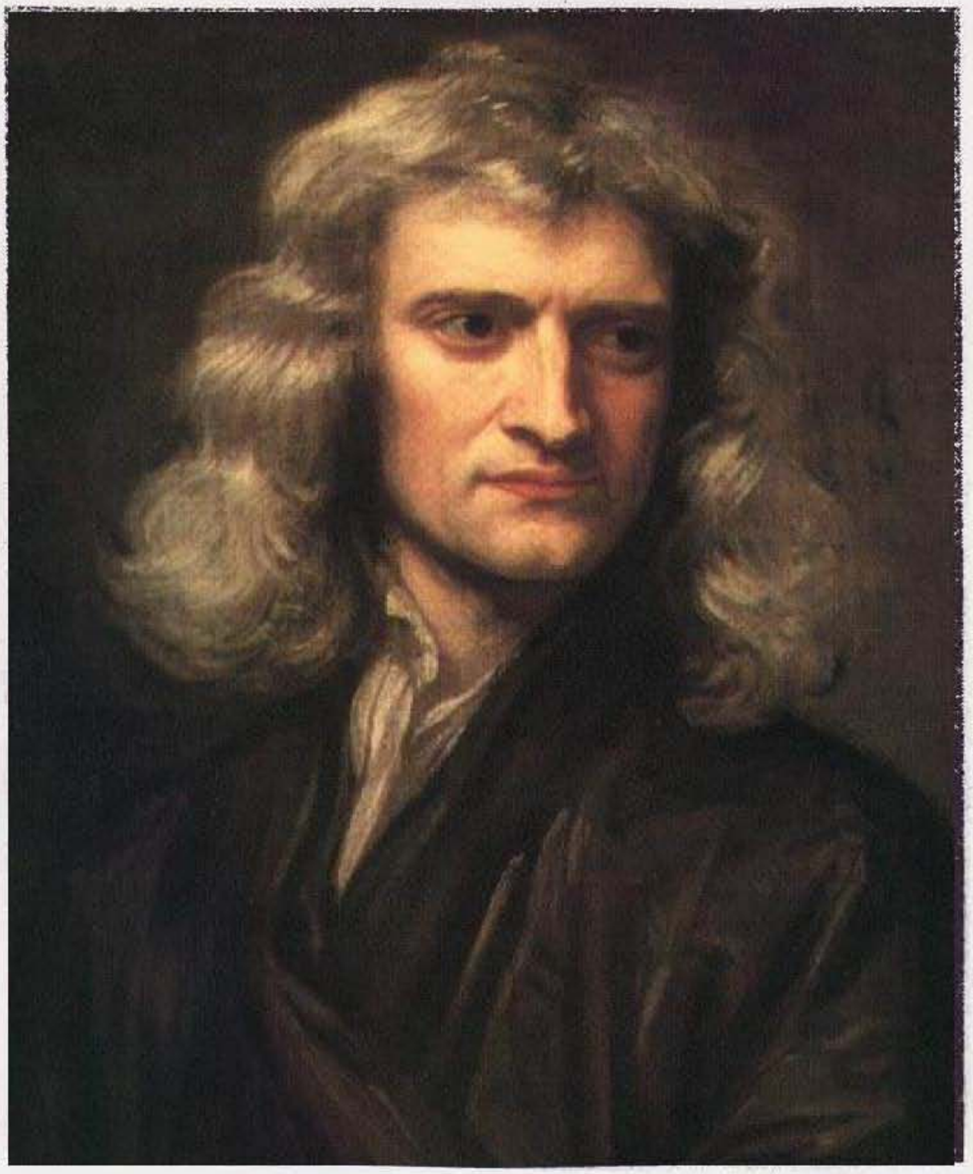
APRIL. 26, 2007

The Next Lecture:

The World of Non-Linear

May 10, 2007

Global Focus on Knowledge Lecture Series



Sir Isaac Newton (1642-1727)

and

Gottfreid Leibnitz (1646-1716)

The Founders of Calculus

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}}$$

$$\bar{v} = \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

at limit of

$\Delta t \to 0$, (instant) velocity at time t

$$v(t) = \lim_{\Delta t \to 0} \bar{v} = \frac{dx(t)}{dt}$$

derivation

$\updownarrow$

$$x(t) = \int_0^t v(\tau)\,d\tau + x(0)$$

integration

The 18th Century: "The Century of Dynamics"

Leonhard Euler (1707-1783)

Euler’ s formula $e^{i\theta} = \cos\theta + i\sin\theta$

Joseph Louis Lagrange (1736-1813)

analytic dynamics

The 19th Century "The Century of Fields"

fluid,
elastic body
heat,
electromagnetic
...

Jean Baptiste Joseph Fourier (1768-1830)

Diffusion Equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

Fourier series

Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Johan Bernoulli (1667-1748)

vibration solution of chord

Jean le Rond d'Alembert (1717-1783)

$$u(t) = f(x+ct) + g(x-ct)$$

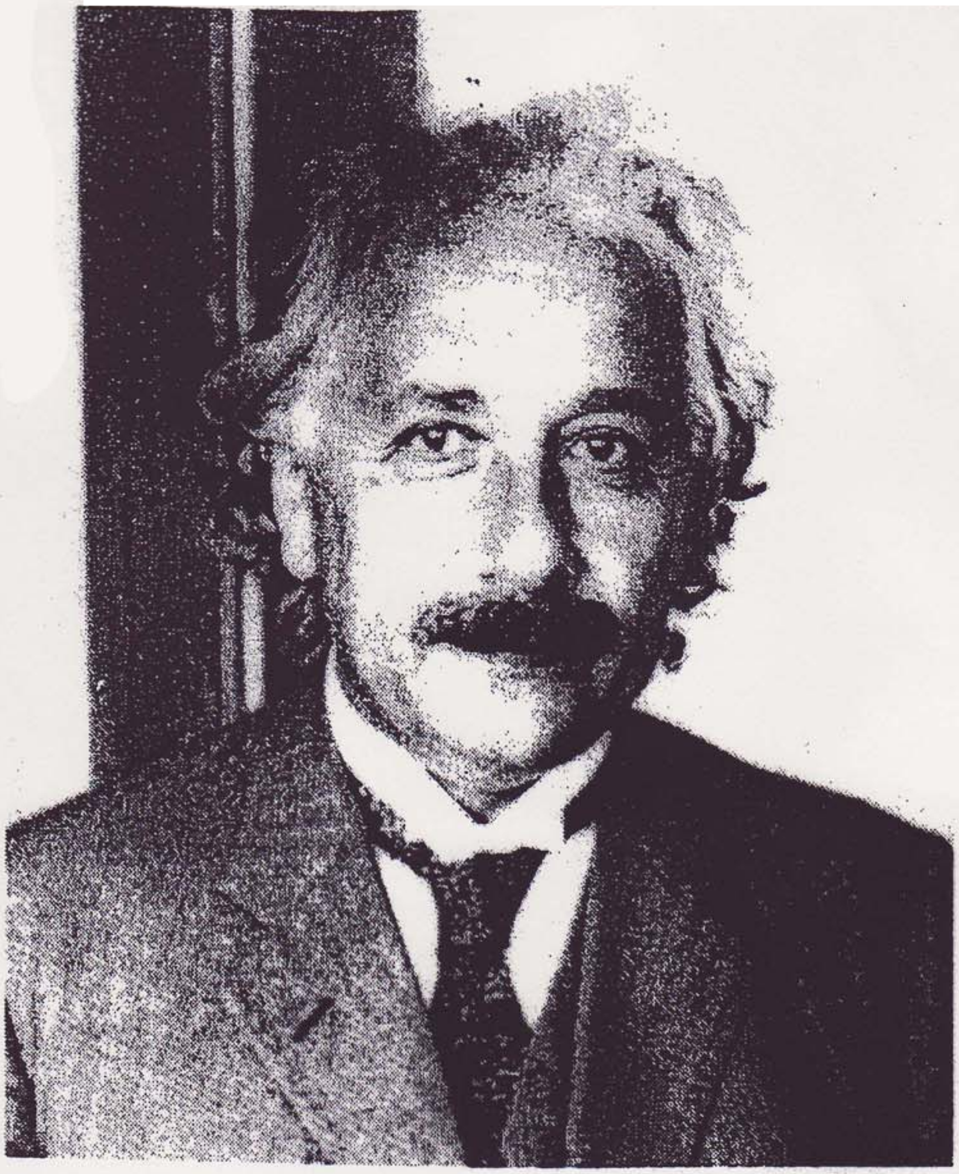
Harmonic Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Laplace (1749-1827)

The 20th Century:

Importance of Algebra and Geometry



Albert Einstein (1879-1955)

light quantum

relativity theory

Brownian motion

To The Non-Linear World

Jules Henri Poincare (1854-1912)

Theories of Dynamics

Sonya Kowalevskaya (1850-1891)

Asymmetric Top

Paul Painleve (1863-1933)

$$\frac{d^2w}{dx^2} = w^2 + x$$

Painlevé transcendental function

Differentiation and Difference

law of Malthus

$u(t)$: number of rats at time t

$$u(t+\Delta t)-u(t)=\alpha\Delta t\, u(t)$$

difference equation

$u(0)=u_0$ initial value

α breeding ratio

solution $u(\Delta t)=(1+\alpha\Delta t)u_0$

$$u(2\Delta t)=(1+\alpha\Delta t)u(\Delta t)$$

$$=(1+\alpha\Delta t)^2 u_0$$

...

$$\underline{u(n\Delta t)=(1+\alpha\Delta t)^n u_0}$$

$\Delta t \to 0$

$$\frac{du(t)}{dt} = \alpha u(t)$$

differential equation

solution

$\tau = n\Delta t$, $n \to \infty$, $\Delta t \to 0$
(finite)

$$u(\tau) = \left\{ \left(1 + \frac{\alpha}{n}\tau\right)^{\frac{n}{\alpha\tau}} \right\}^{\alpha\tau} u_0$$

$$\to \underline{e^{\alpha\tau} u_0}$$

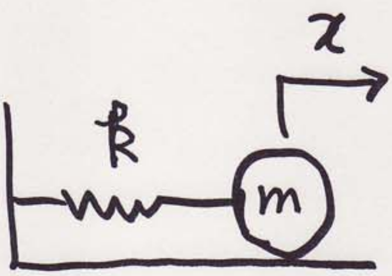
Rem. $\left(\frac{d}{dt} - \alpha\right) u = 0$

the solution is $u = u_0 e^{\alpha t}$

Newton's Laws of Motion

$$m\frac{d^2x}{dt^2} = F$$

ex.



$$m\frac{d^2x}{dt^2} + kx = 0$$

or

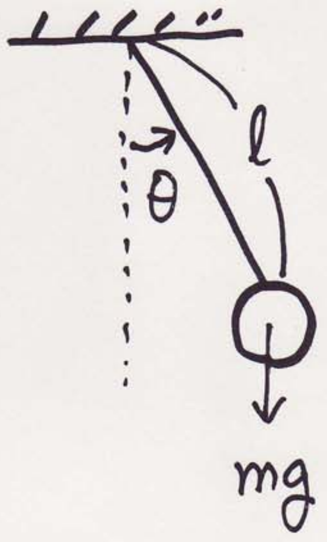
$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$$

solution

From $\left(\frac{d}{dt} + i\omega\right)\left(\frac{d}{dt} - i\omega\right)x = 0$

$$x = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$$
$$= \tilde{c}_1 \cos\omega t + \tilde{c}_2 \sin\omega t$$

ex. 2



$$m\ell \frac{d^2\theta}{dt^2} = -mg\sin\theta$$

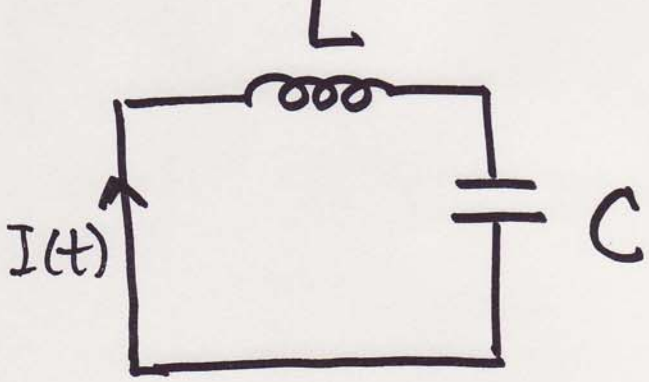
or

$$\frac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0$$

When $\theta \ll 1$

$$\frac{d^2\theta}{dt^2} + \omega^2\theta = 0$$

ex. 3



$$\frac{d^2 I}{dt^2} + \frac{1}{LC} I = 0$$

equivalent circuit

These three examples are expressed in
a same differential equation.

$$\frac{d^2 x}{dt^2} + \omega^2 x = 0$$

Characteristics of Linear Equations

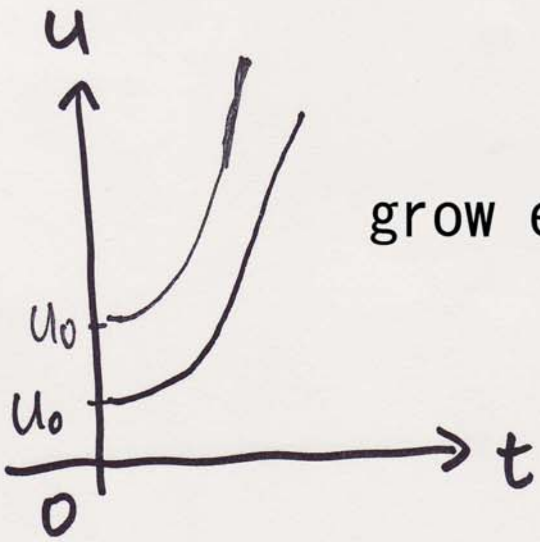
1) Solutions can be superposed.

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

solutions: $x = e^{i\omega t}$, $e^{-i\omega t}$

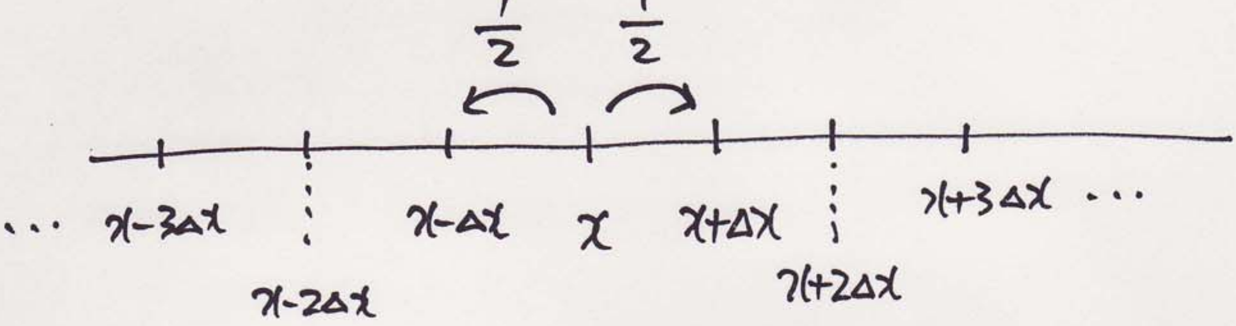
2) Initial values do not determine how solutions behave.

$\frac{du}{dt} = \alpha u$ solution: $u = u_0 e^{\alpha t}$



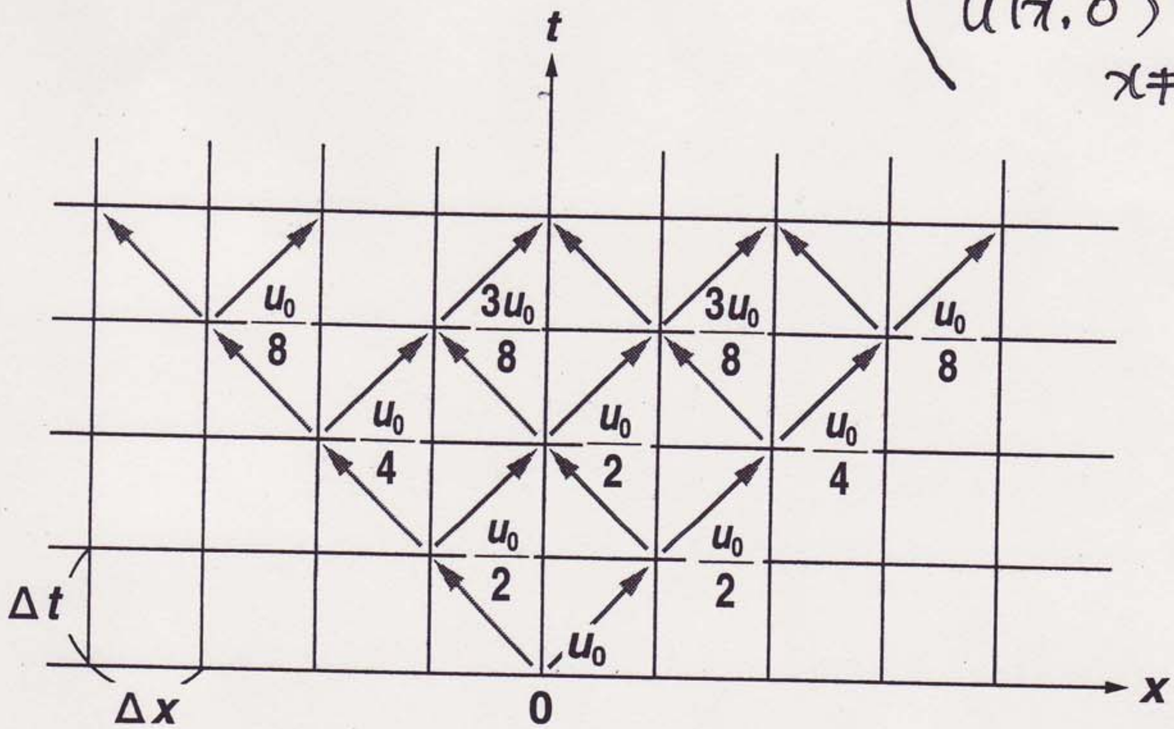
grow exponentially

Random Walk



$$u(x, t+\Delta t) = \frac{1}{2}\left\{ u(x-\Delta x, t) + u(x+\Delta x, t) \right\}$$

initial value: $\begin{cases} u(0,0) = u_0 \\ u(x,0) = 0 \quad x \neq 0 \end{cases}$



develops in binomial distribution

continuous limit

Taylor expansion

$$u(x,t+\Delta t) = u(x,t) + \frac{\partial u(x,t)}{\partial t}\Delta t + \cdots$$

$$u(x \pm \Delta x, t) = u \pm \frac{\partial u}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(\Delta x)^2 + \cdots$$

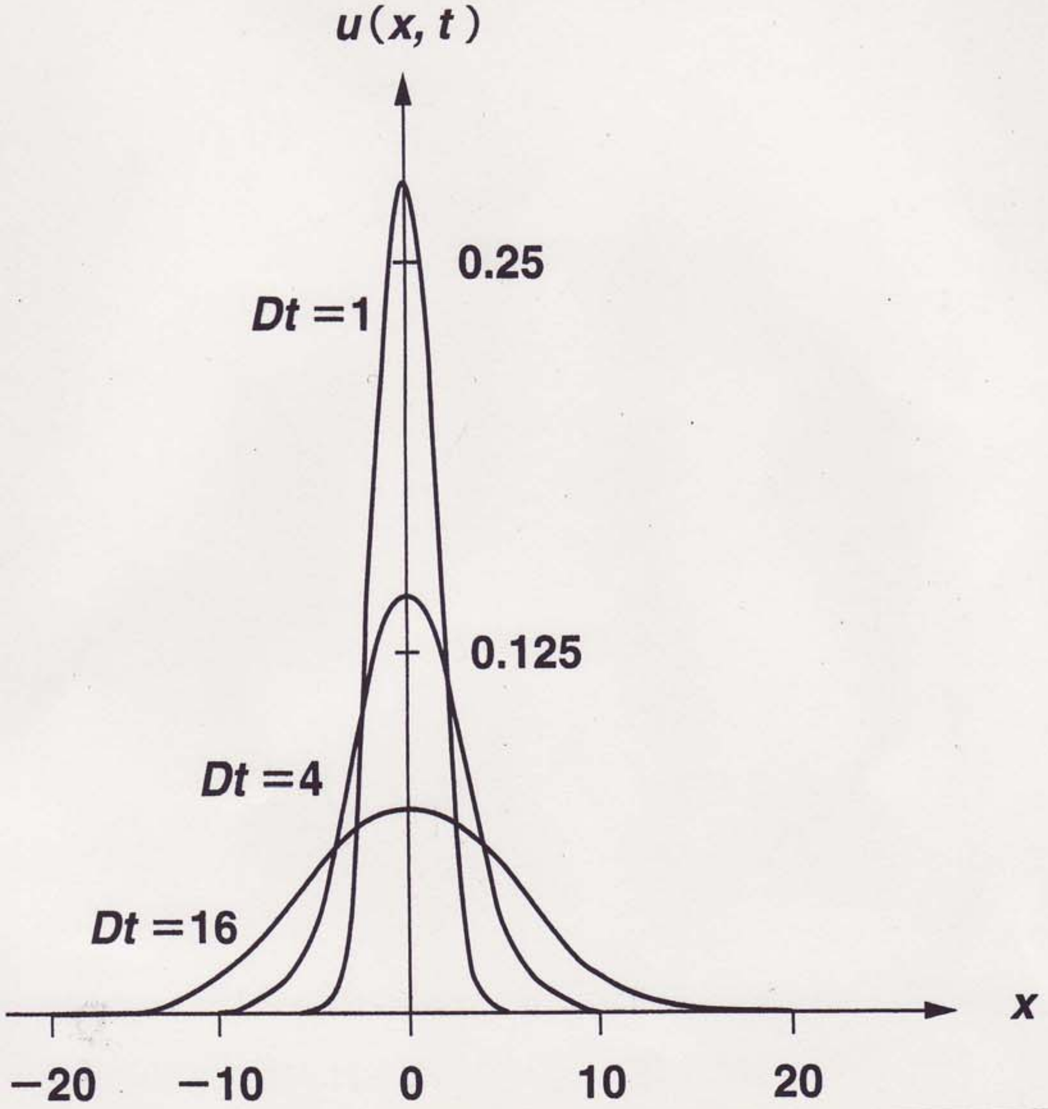
when $\Delta x,\ \Delta t \to 0$

however, $\frac{(\Delta x)^2}{\Delta t} = 2D$ (constant)

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

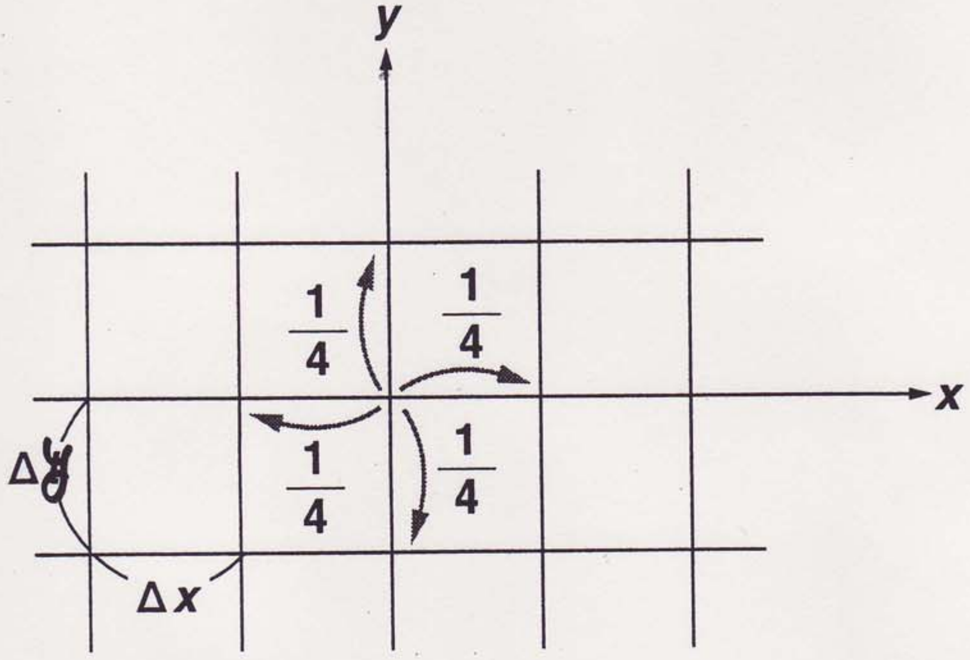
diffusion equation

Solution to Diffusion Equation



Normal Distribution

2-dimensional Random Walk



$$u(x, y, t+\Delta t) = \frac{1}{4}\left\{ u(x-\Delta x, y, t) + u(x+\Delta x, y, t) + u(x, y-\Delta y, t) + u(x, y+\Delta y, t) \right\}$$

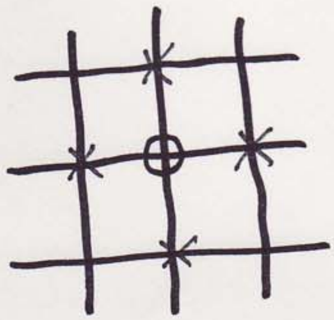
$$\Downarrow$$

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

2-D diffusion equation

When u is Not Dependent on t,

$$u(x,y) = \frac{1}{4}\left\{ u(x-\Delta x, y) + u(x+\Delta x, y) + u(x, y-\Delta y) + u(x, y+\Delta y) \right\}$$



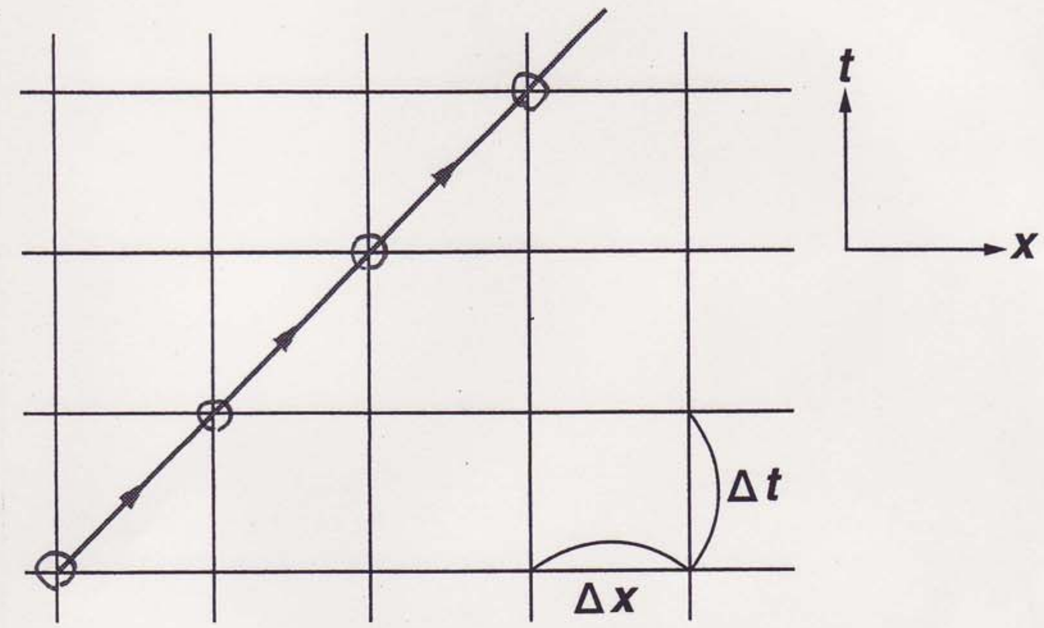
continuous limit

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

harmonic equation

Wave System

$$u(x, t+\Delta t) = u(x-\Delta x, t)$$



when $\Delta x, \Delta t \to 0$

however, $\dfrac{\Delta x}{\Delta t} = c$ (constant)

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

wave equation

solution: $u(x,t) = f(x-ct)$

左右に伝わる波

$$u(x, t+\Delta t) + u(x, t-\Delta x)$$

$$= u(x+\Delta x, t) + u(x-\Delta x, t)$$

連続極限

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

波動方程式