The World of Non-Linear

Junkichi Satsuma

Date. May 10, 2007

Global Focus on Knowledge Lecture Series,
the University of Tokyo


冨嶽三十六景
神奈川沖
浪裏

Logistic eq.

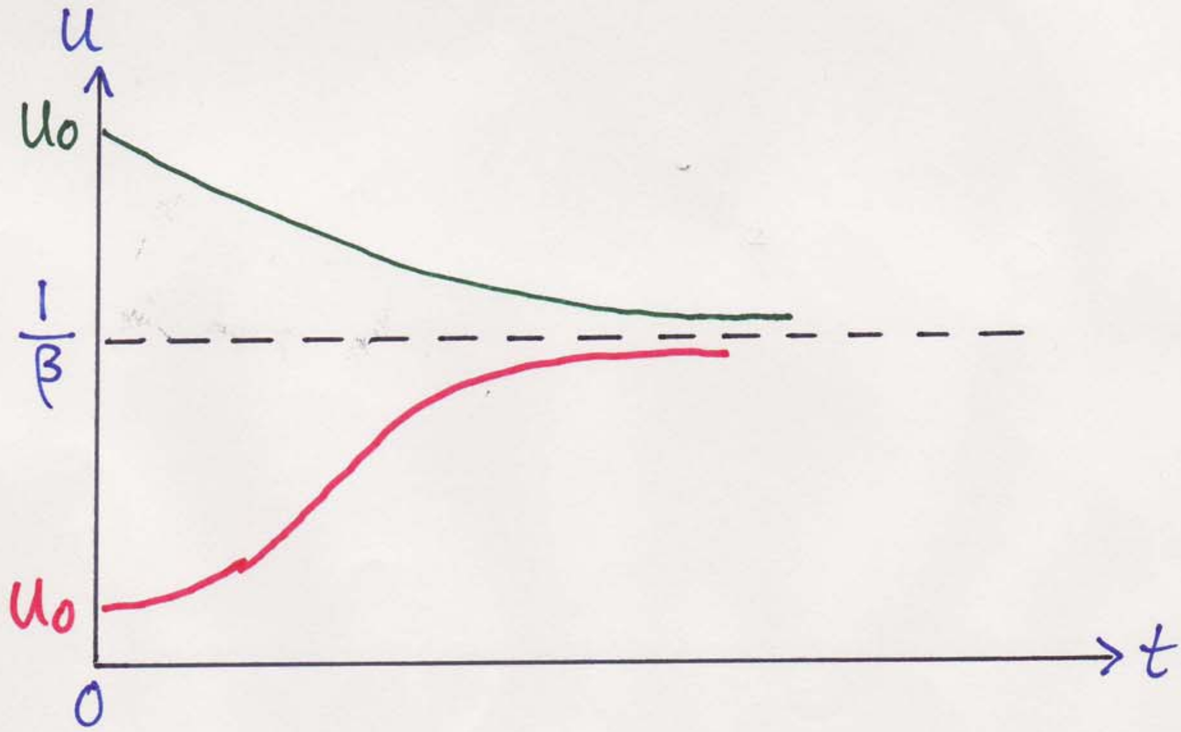
$$\frac{du}{dt} = \alpha(1-\beta u)u$$

$u(0) = u_0$ initial value

α breeding ratio $\quad$ β : crowdness constant

solution

$$u(t) = \frac{u_0}{\beta u_0 + (1-\beta u_0)e^{-\alpha t}}$$



Simple Dfferentiation

$$u(t+\Delta t) - u(t)$$

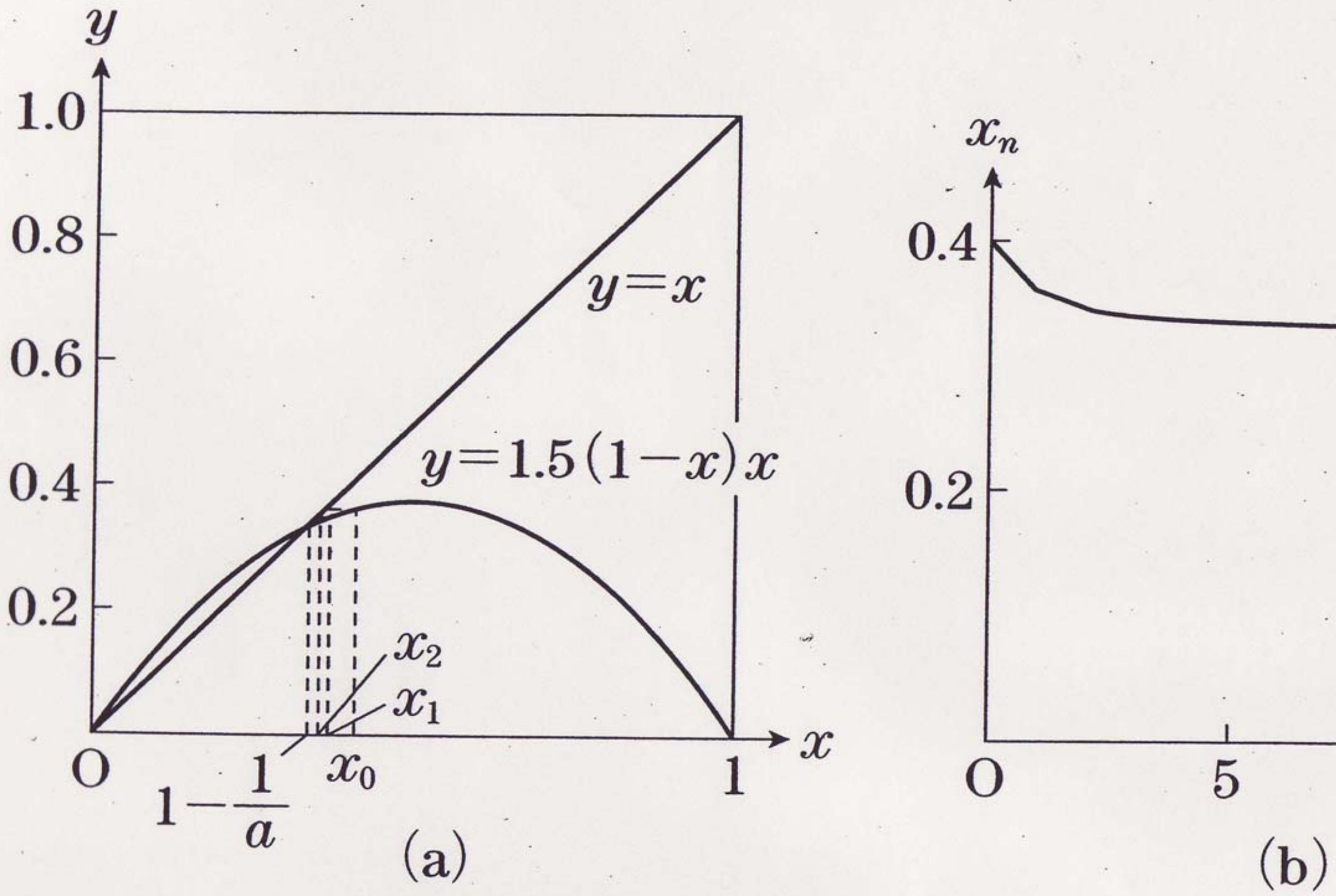
$$= \alpha \Delta t \{1 - \beta u(t)\} u(t)$$

$$x_n = \frac{\alpha \beta \Delta t}{1+\alpha \Delta t} u(n\Delta t)$$

$$a = 1 + \alpha \Delta t, \quad t = n \Delta t$$

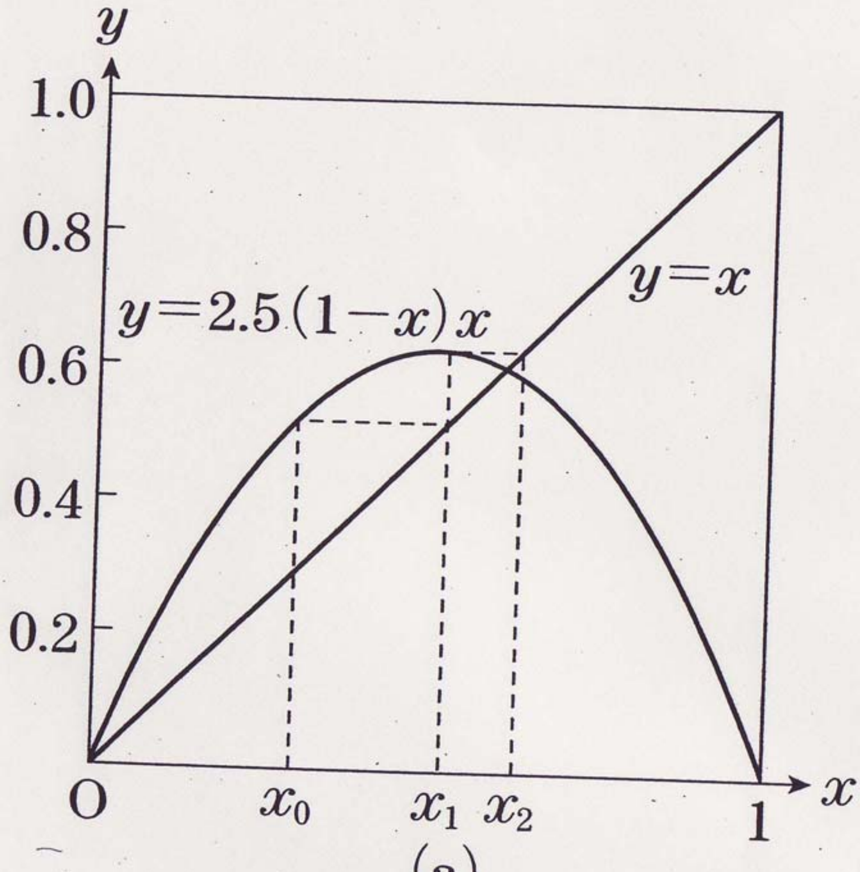
$$x_{n+1} = a(1-x_n)x_n$$

logistic

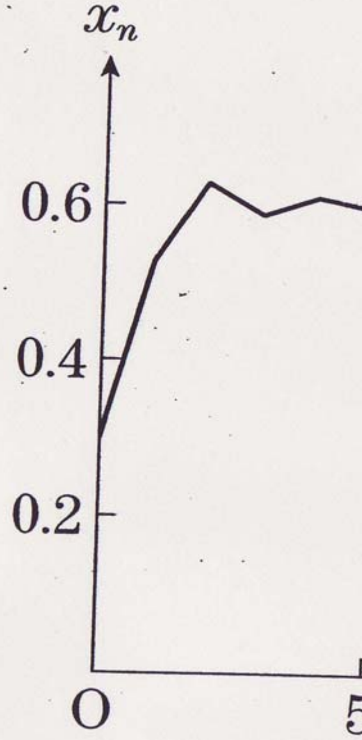


Graph of $y = f_a(x)$ and changes of x_n $(a$

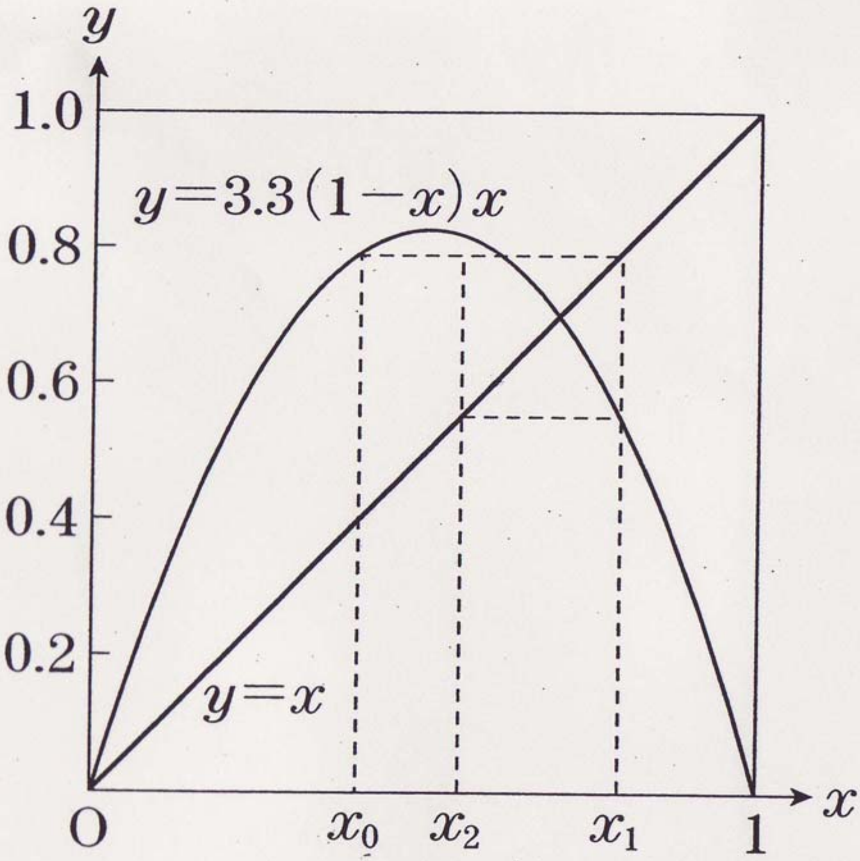
a= 1-

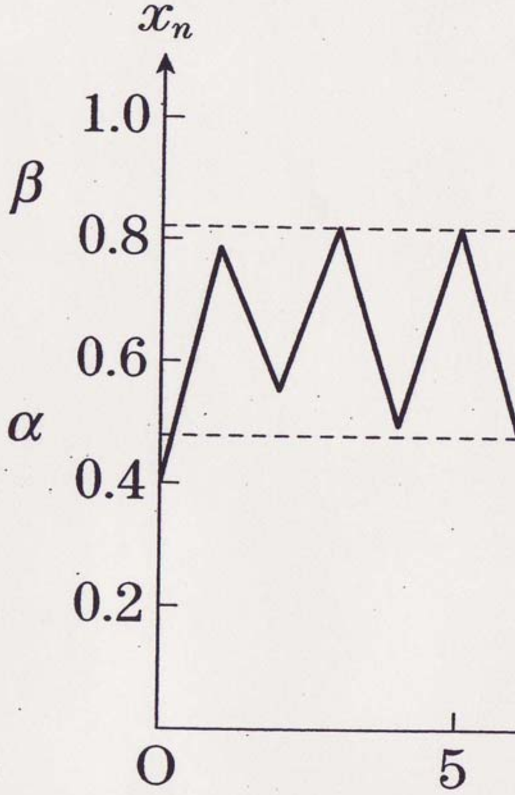


(a)

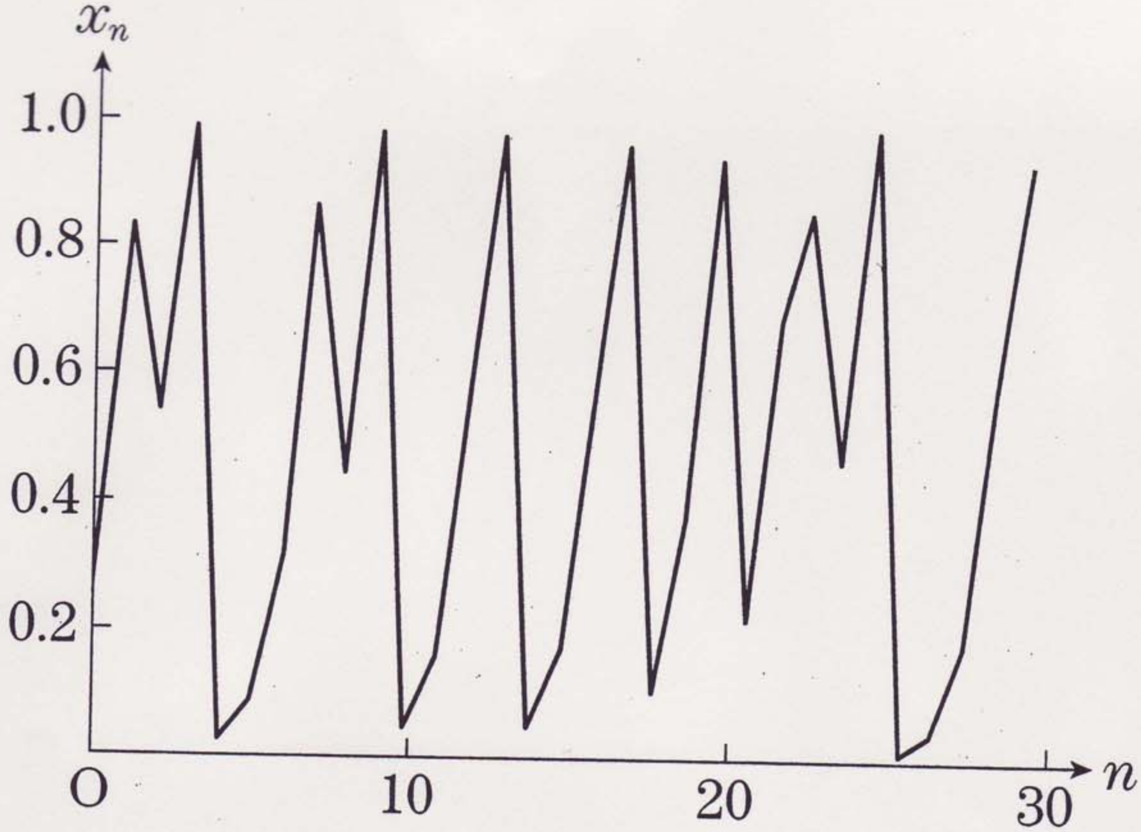


Graph of $y = f_a(x)$ changes of x_n





Graph of $y = f_a(x)$ changes of x_n $(a$



chaos state $(a = 4.0,\ x_0 = 0.3)$

According to

- x_0 , orbit with any period can be made.

  There will be an orbit without period also.

- With a little change in X_0, an orbit might change much.

0

May (1974)

Japanese Atractor

Ueda (1973 (1961))

Duffin eq.

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + x^3 = B\cos t$$

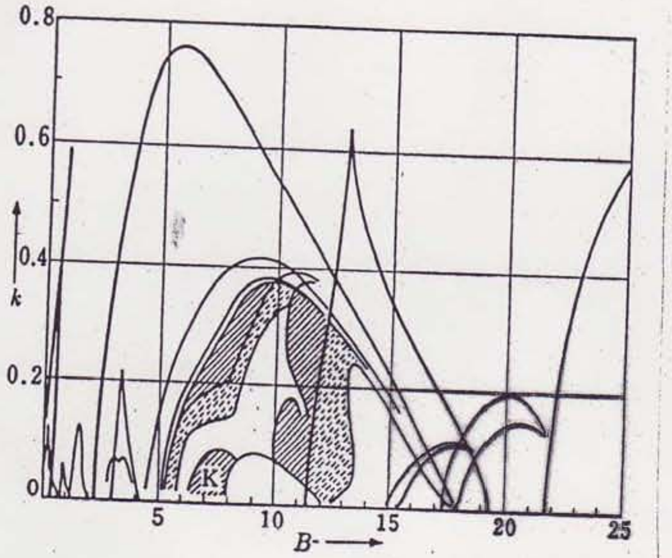
$$\Big\downarrow \quad \frac{dx}{dt} = y$$

$$\begin{cases} \dfrac{dy}{dt} = -ky - x^3 + B\cos t \\[2ex] \dfrac{dx}{dt} = y \end{cases}$$

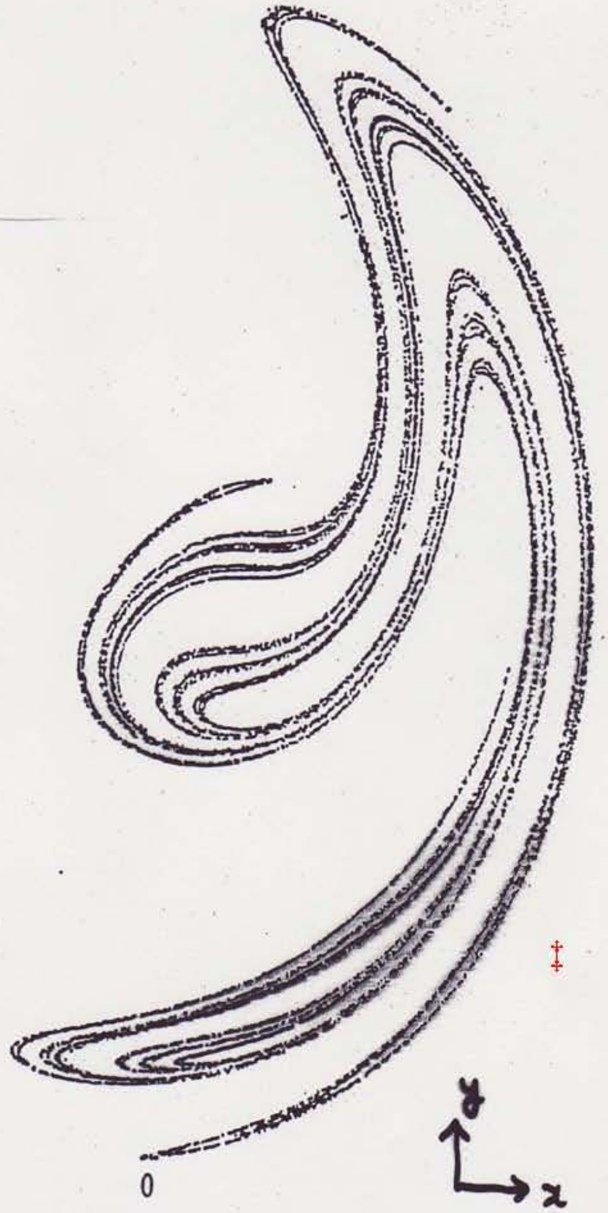
$[x(0), y(0)] \rightarrow [x(2\pi), y(2\pi)] \rightarrow$

$\cdots \rightarrow [x(2n\pi), y(2n\pi)]$

2-DE mapping

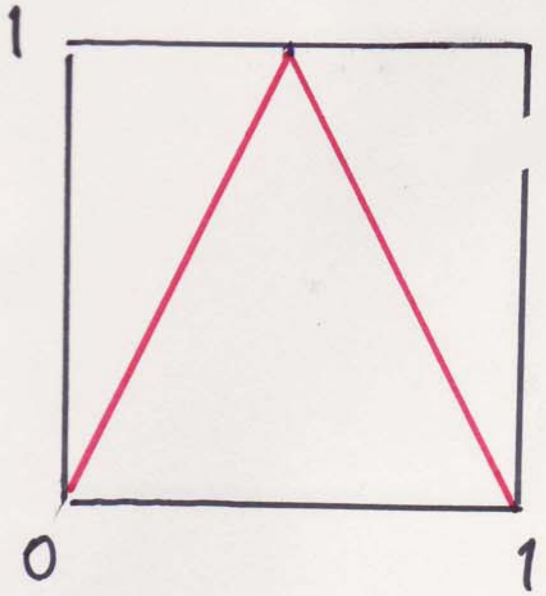


When k and B is in a shaded area in the above graph, the orbit converges to the Japanese atractor at right.



Japanese atractor

Chaos and Fractal



tent-type image

$$x_{n+1} = f(x_n) = \begin{cases} & 0 < x_n < 0.5 \\ 2(1-x_n) & 0.5 < x_n < 1 \end{cases}$$

$$x_1 = f(x_0),\ x_2 = f(x_1) = f^2(x_0), \cdots, x_n = f^n(x_0), \cdots$$

$x_1, x_2, \cdots$ an irregular sequence appears.

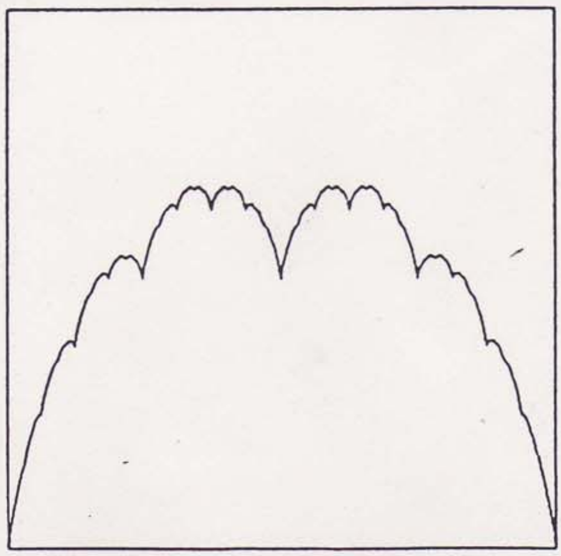
chaos

$$T(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f^n(x)$$

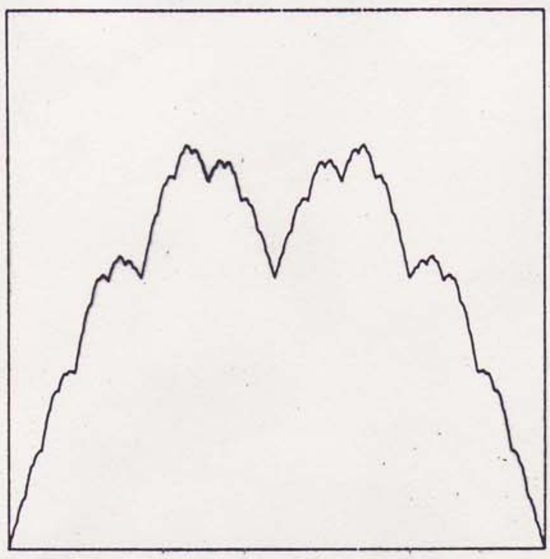
Takagi function

Continuous, but unable to differentiate at many spots.

Takagi Function



Weierstrass Function

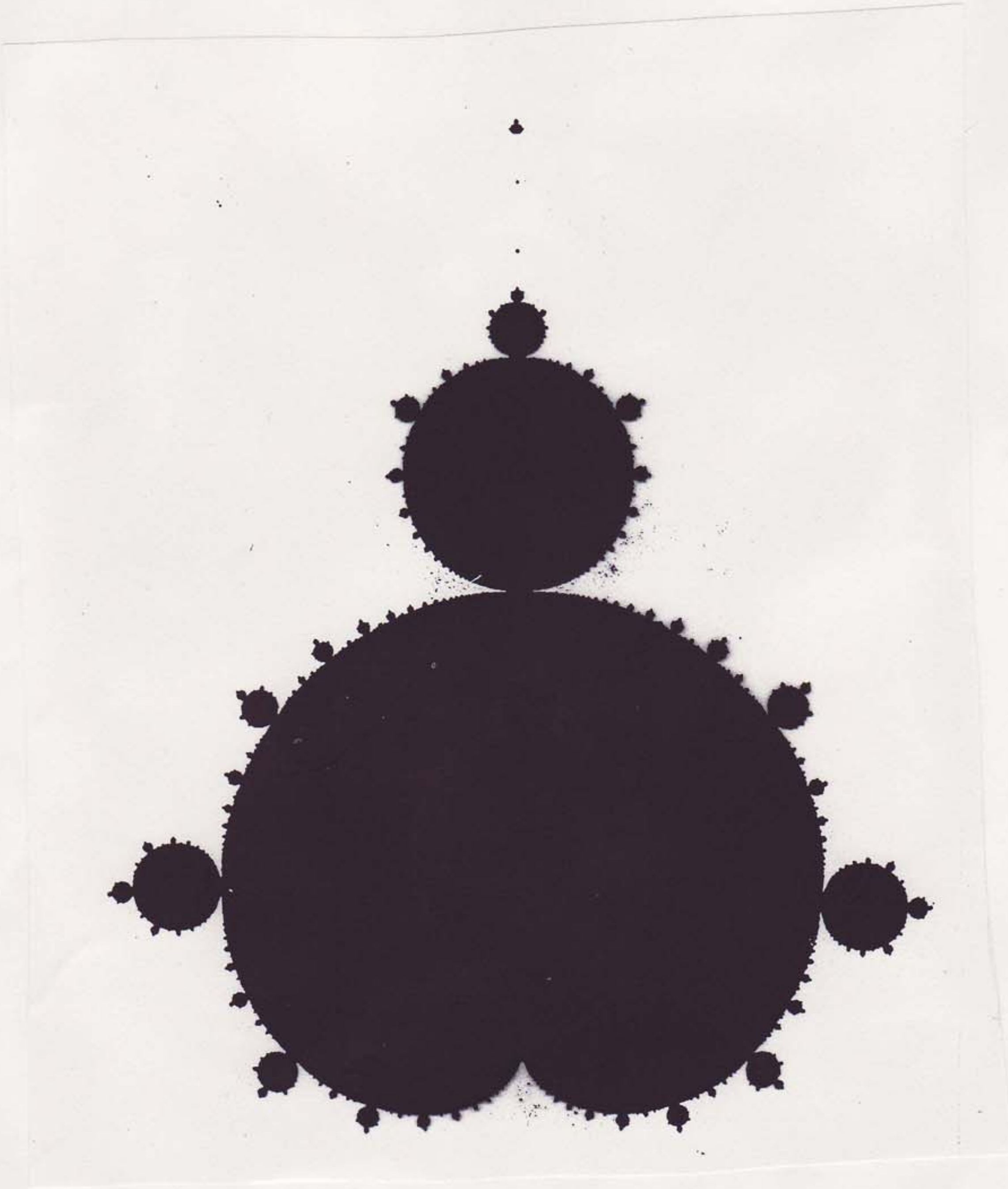


$$W(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} \cos\left(\pi f^{(n)}(x)\right)$$

Mandelbrot

$$Z_{n+1} = f(Z_n) = Z_n^2 + \mu \quad , \quad Z_n = x_n + iy_n$$

set of μ having Z_0 as an origin, and when $n \to \infty$, $|Z_n| \to \infty$



Logistic eq. the Other Difference Method

$$u(t+\Delta t) - u(t)$$

$$= \alpha \Delta t \{1 - \beta u(t)\} u(\underline{t+\Delta t})$$

$$W(t) = \frac{1-\beta u(t)}{u(t)}$$ möbius conversion

$$W(t+\Delta t) - W(t) = -\alpha \Delta t \, W(t)$$

solution

$$W(n\Delta t) = (1-\alpha\Delta t)^n W(0)$$

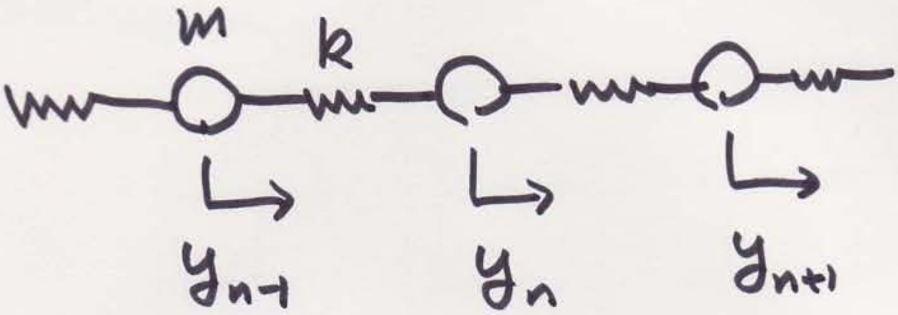
OR

$$u(n\Delta t) = \frac{u_0}{\beta u_0 + (1-\beta u_0)(1-\alpha\Delta t)^n}$$

$$u_0 = u(0)$$

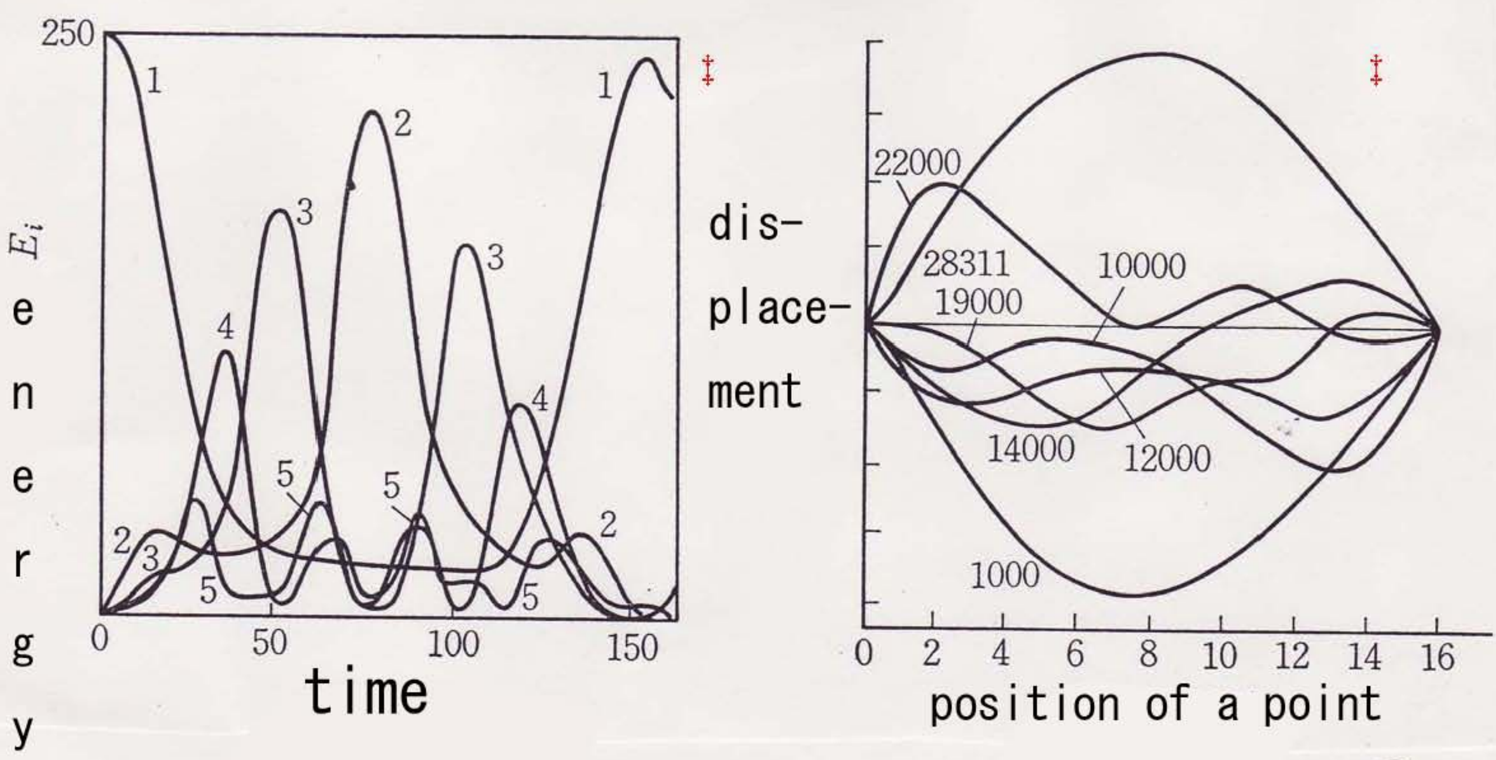
Fermi, Pasta, Ulam Problem

(1950s)



$$m\frac{d^2y_n}{dt^2} = k\{(y_{n+1}-y_n)+\alpha(y_{n+1}-y_n)^2\}$$
$$-k\{(y_n-y_{n-1})+\alpha(y_n-y_{n-1})^2\}$$

Recurrence Phenomenon



Zabusky, Kruskal (1965)

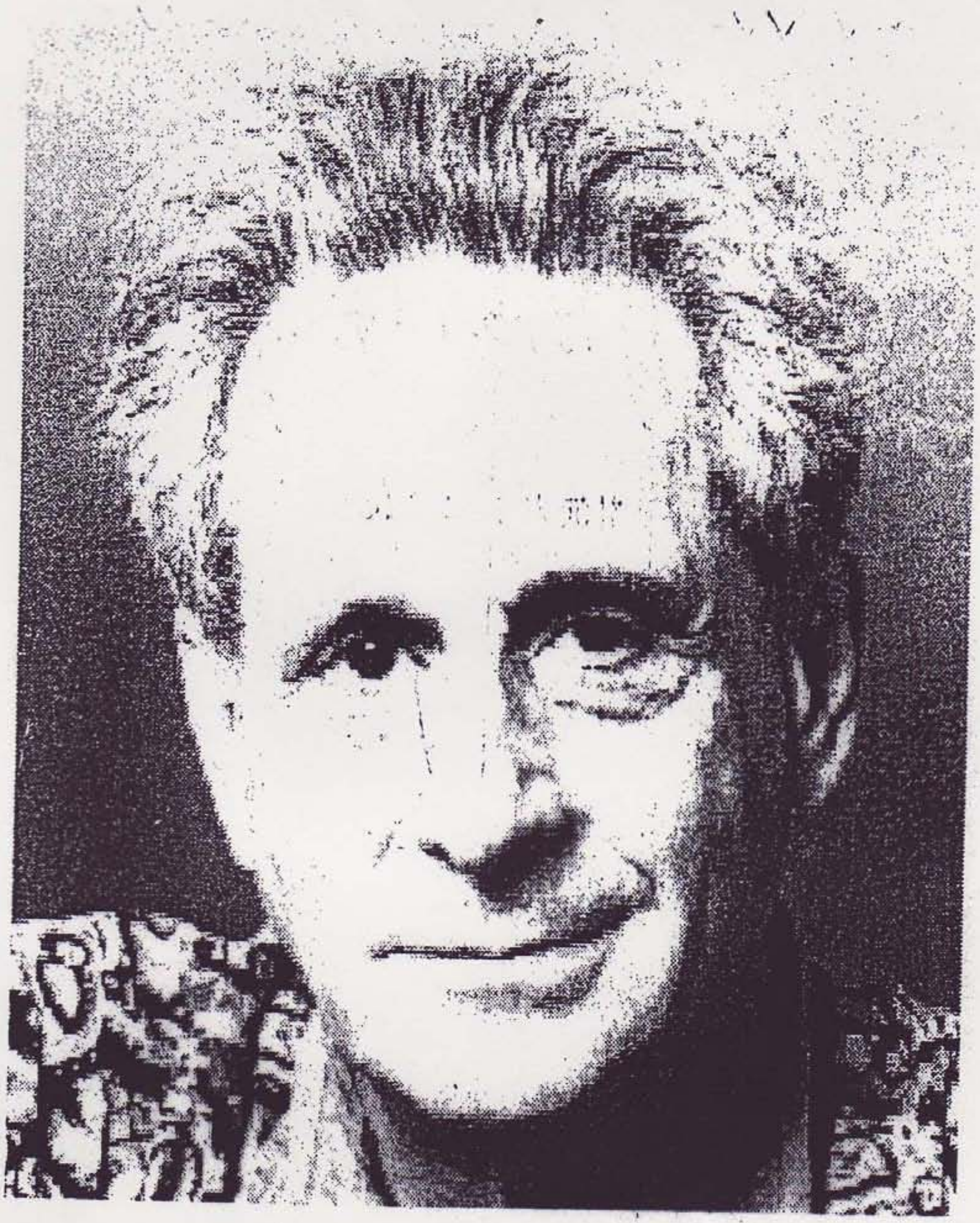
Lattice by Fermi et al. was approximated and KdV equation was calculated

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

Discovery of

Soliton (Solitary-ton)

Figure removed due to copyright restrictions



Martin Kruskal (1925-2006. Dec.26)

Miura conversion (1967)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

$$u = v^2 \pm \sqrt{6}\, i \frac{\partial v}{\partial x}$$

Riccati equation

$$\frac{\partial v}{\partial t} + v^2\frac{\partial v}{\partial x} + \frac{\partial^3 v}{\partial x^3} = 0$$

- existence of infinite conservative quantity

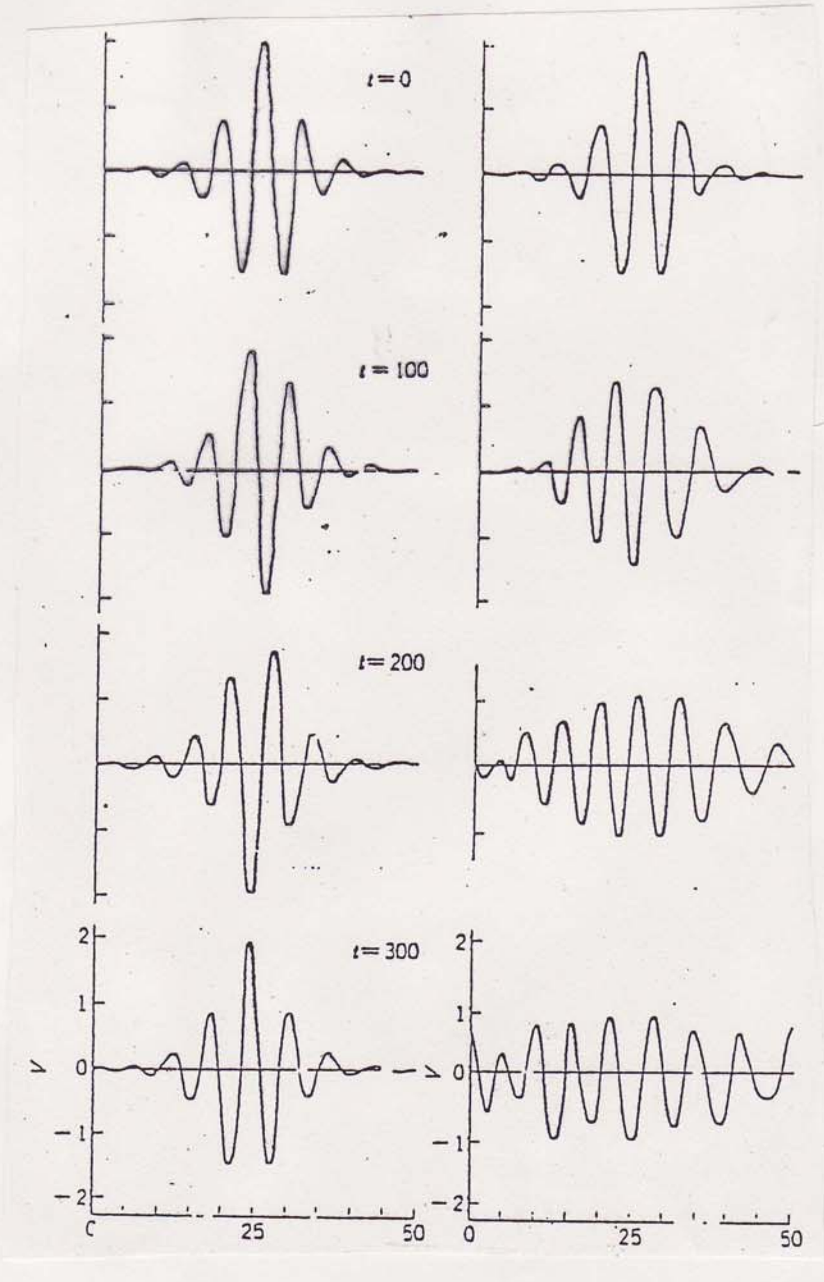
- Inverse scattering method which solves initial value problem strictly was discovered.

Gardner, Greene, Kruskal, Miura

Non-Linear Schrödinger Equation

$$i \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + |u|^2 u$$

non-linear linear



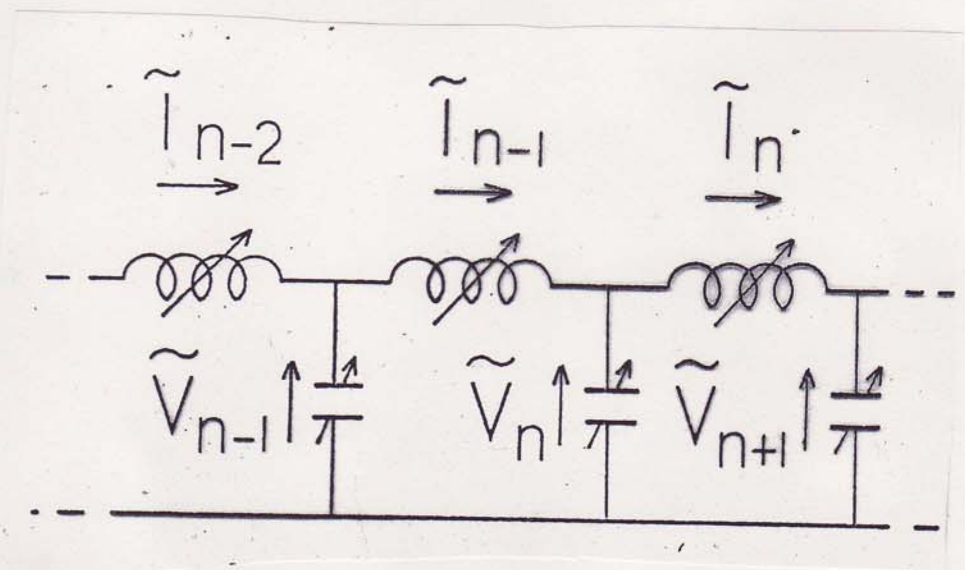
Toda Lattice (1967)

$$\frac{d^2 y_n}{dt^2} = e^{-y_n + y_{n-1}} - e^{-y_{n+1} + y_n}$$

$$\updownarrow \quad V_n = e^{-y_n + y_{n-1}} - 1$$

$$\frac{d}{dt} \log (1 + V_n) = I_{n-1} - I_n$$

$$\frac{d}{dt} I_n = V_n - V_{n+1}$$



Kadomtsev-Petviashvili (KP) eq.

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) + \alpha \frac{\partial^2 u}{\partial y^2} = 0$$

Figure removed due to copyright restrictions.

Expansion to infinite dimensional integrability

- Hirota's method
- τ function theory

⋮

Hirota/Miwa Equation

$$
\begin{aligned}
&\tau_n(\ell+1, m+1)\,\tau_n(\ell, m) \\
&- \tau_n(\ell+1, m)\,\tau_n(\ell, m+1) \\
&= ab\Big\{ \tau_{n+1}(\ell, m+1)\,\tau_{n-1}(\ell+1, m) \\
&\qquad - \tau_n(\ell+1, m+1)\,\tau_n(\ell, m) \Big\}
\end{aligned}
$$

Ultra Discrete Series

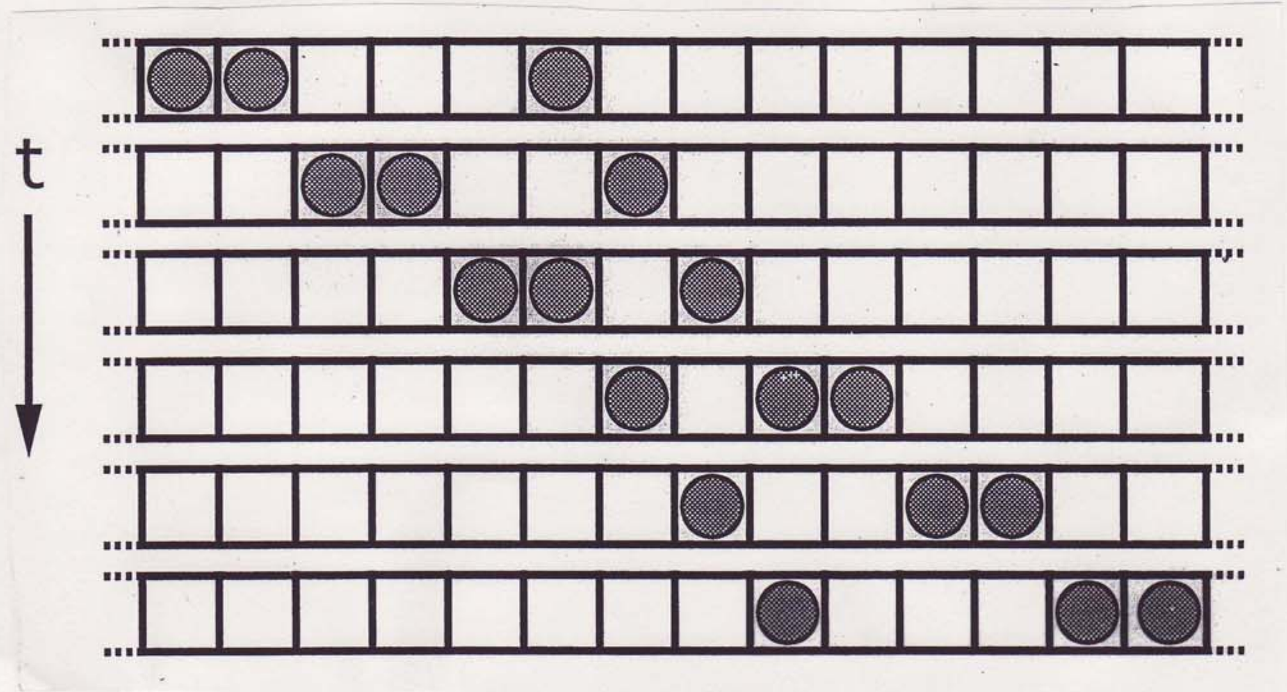
Soliton Cellular Automaton

n →

t ↓

...01100010000000...
...00011001000000...
...00000110100000...
...00000001011000...
...00000000100110...

box-ball system



Ultra Discretization Limit

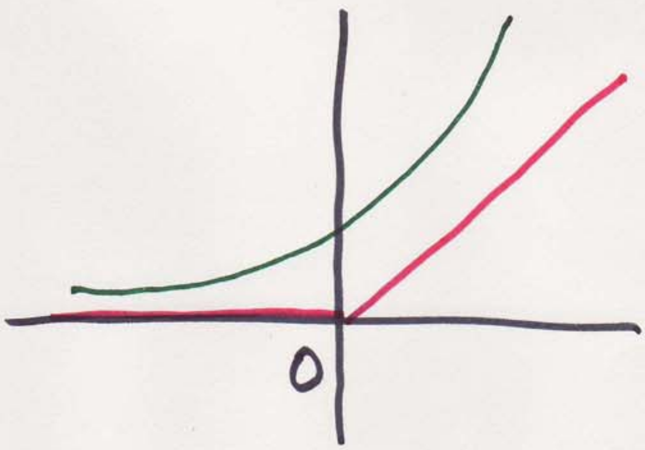
differentiation eq. KdV eq.

continuous limit

difference eq. Lotka-Volterra type difference eq.

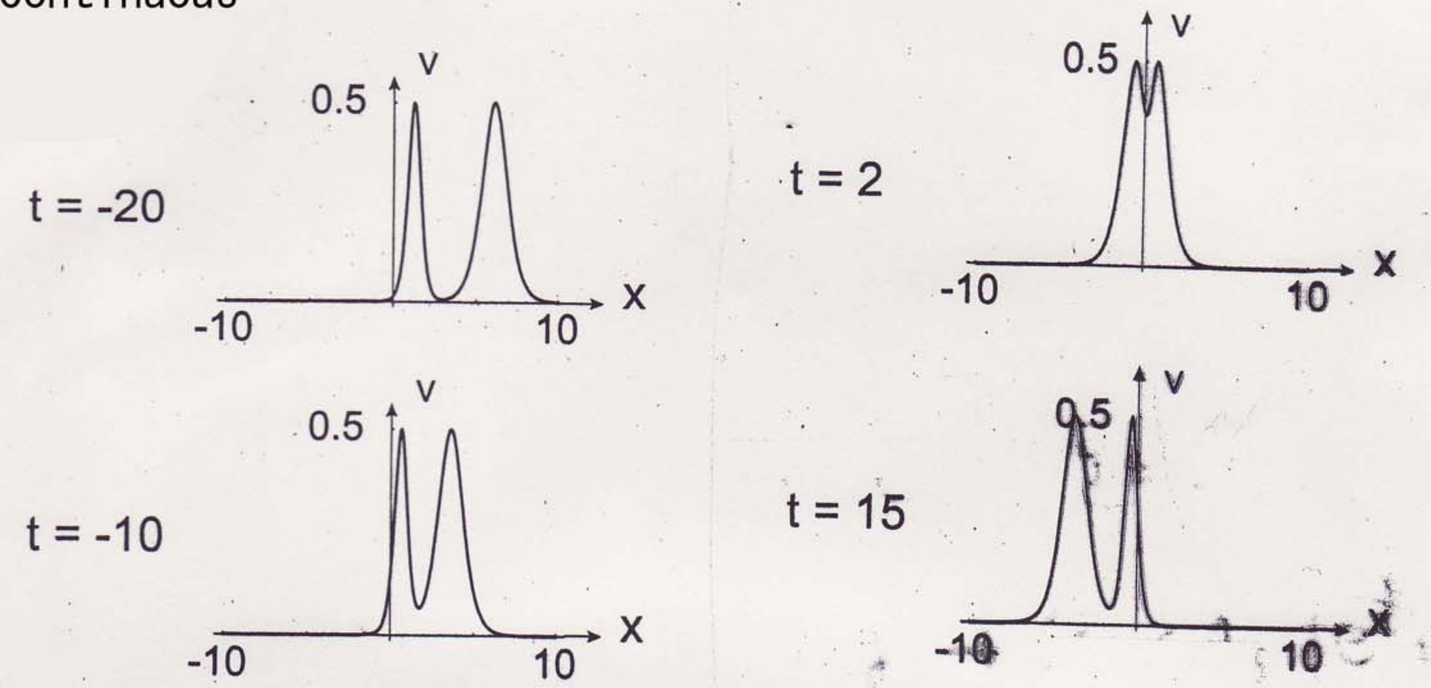
ultra discretization limit

$$\lim_{\varepsilon \to +0} \varepsilon \log\left(e^{\frac{a}{\varepsilon}} + e^{\frac{b}{\varepsilon}}\right) = \max(a, b)$$



ultra discrete series box-ball system

continuous



ultra discretization