



Global Focus on Knowledge

“Creating mathematics” lecture 11

Shapes of Polyhedrons

and Shapes of Orbits on the Curved Surface

Graduate school of mathematical sciences

Takashi Tsuboi

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教科書にはのっていない数学のお話

主題科目 / テーマ講義 2単位 1、2年生対象
数学を創る—数学者達の挑戦—

コーディネータ・ナビゲータ:岡本和夫(理学部)



数学はどうやって創られたか 岡本和夫(理学部)、室田一雄(工学部)

第1回 10/8 数学はどうやって創られたか



ことばを創り、世界を創る 斎藤毅(理学部)

第2回 10/15 Mathematics "On Campus"

第3回 10/22 数の体系を創る



第4回 10/29 数と図形の共進化

脳と情報の数学を創る 甘利俊一(理化学研究所)

第5回 11/5 情報の仕組み:驚き、確率、幾何学

第6回 11/12 脳の仕組み:脳内情報の表現、記憶、学習の数理



目の錯覚の数学を創る 新井仁之(理学部)

第7回 11/19 数学で探る錯視の世界

第8回 11/26 脳の中のウェーブレット

第9回 12/3 錯視が創る新たな数学—ウェーブレットからフレームレットへ—



形を理解するための数学を創る 坪井俊(理学部)

第10回 12/10 惑星の軌道を理解する

第11回 12/17 多面体の形と曲面上の軌道の形

第12回 1/14 形の見分け方と数学の視点



文化と数学 岡本和夫(理学部)

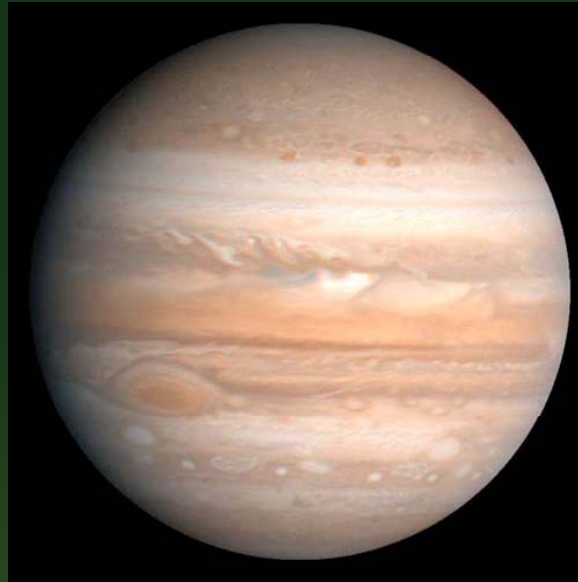
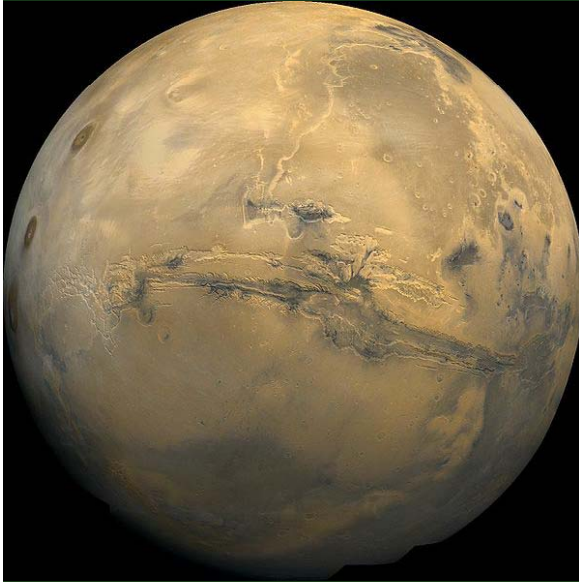
第13回 1/21 文化と数学

駒場 キャンパス 18号館ホール 木曜日 5時限 (16:20-17:50)

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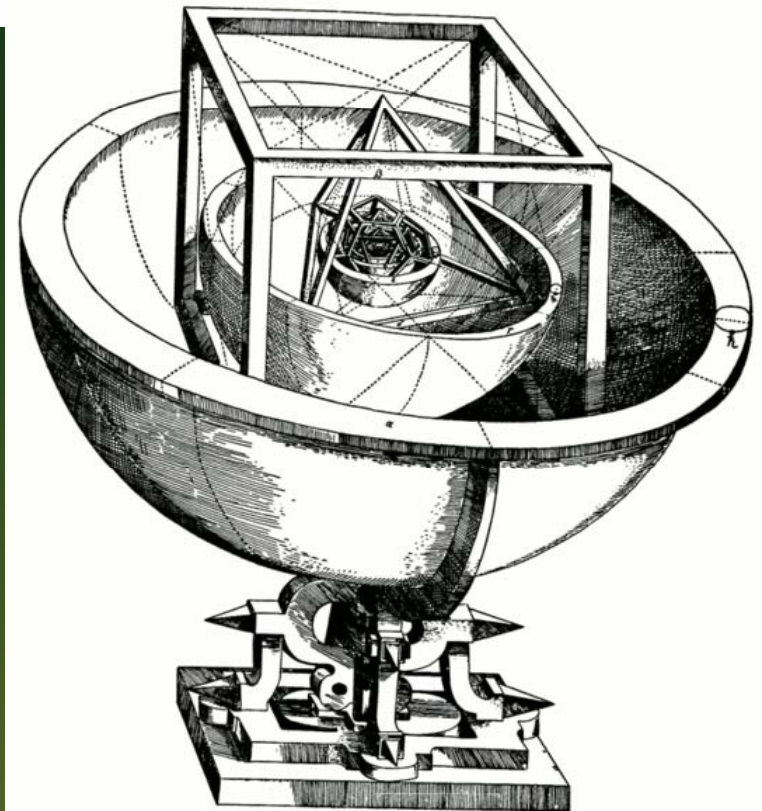
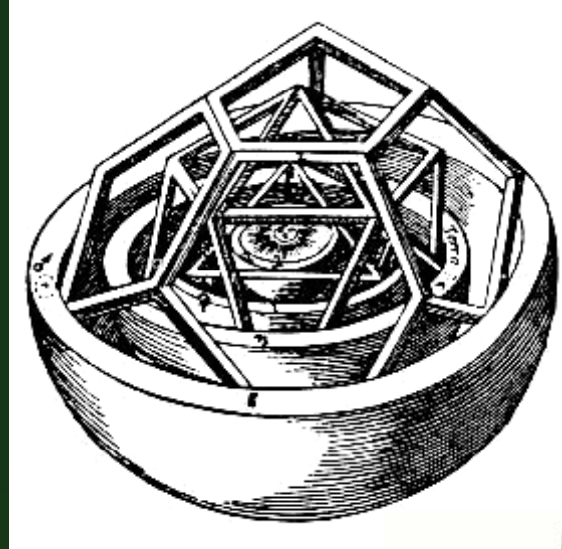


planets

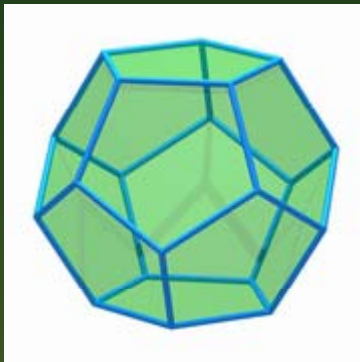
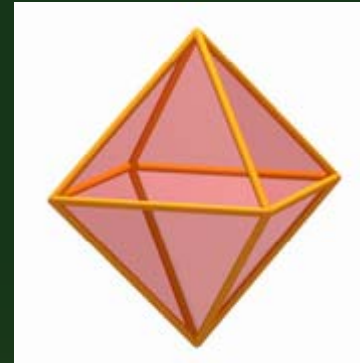
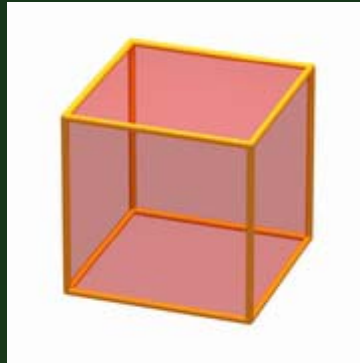
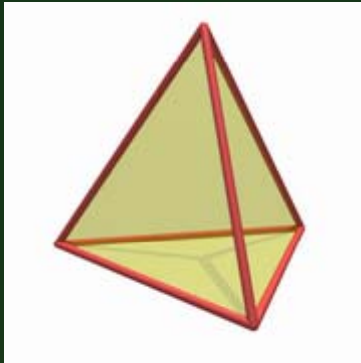


Kepler(1571—1630)

Kepler formulated three laws of planetary motion, but on the other hand, he recklessly tried to find a relation between the distance between the sun and six planets and the five regular polyhedrons.



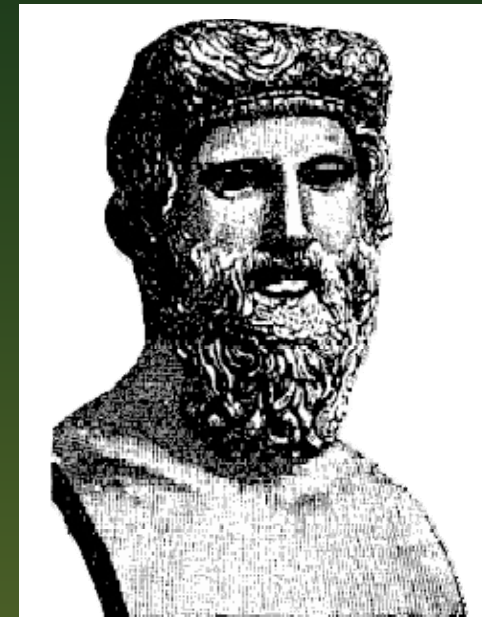
Platonic Solids



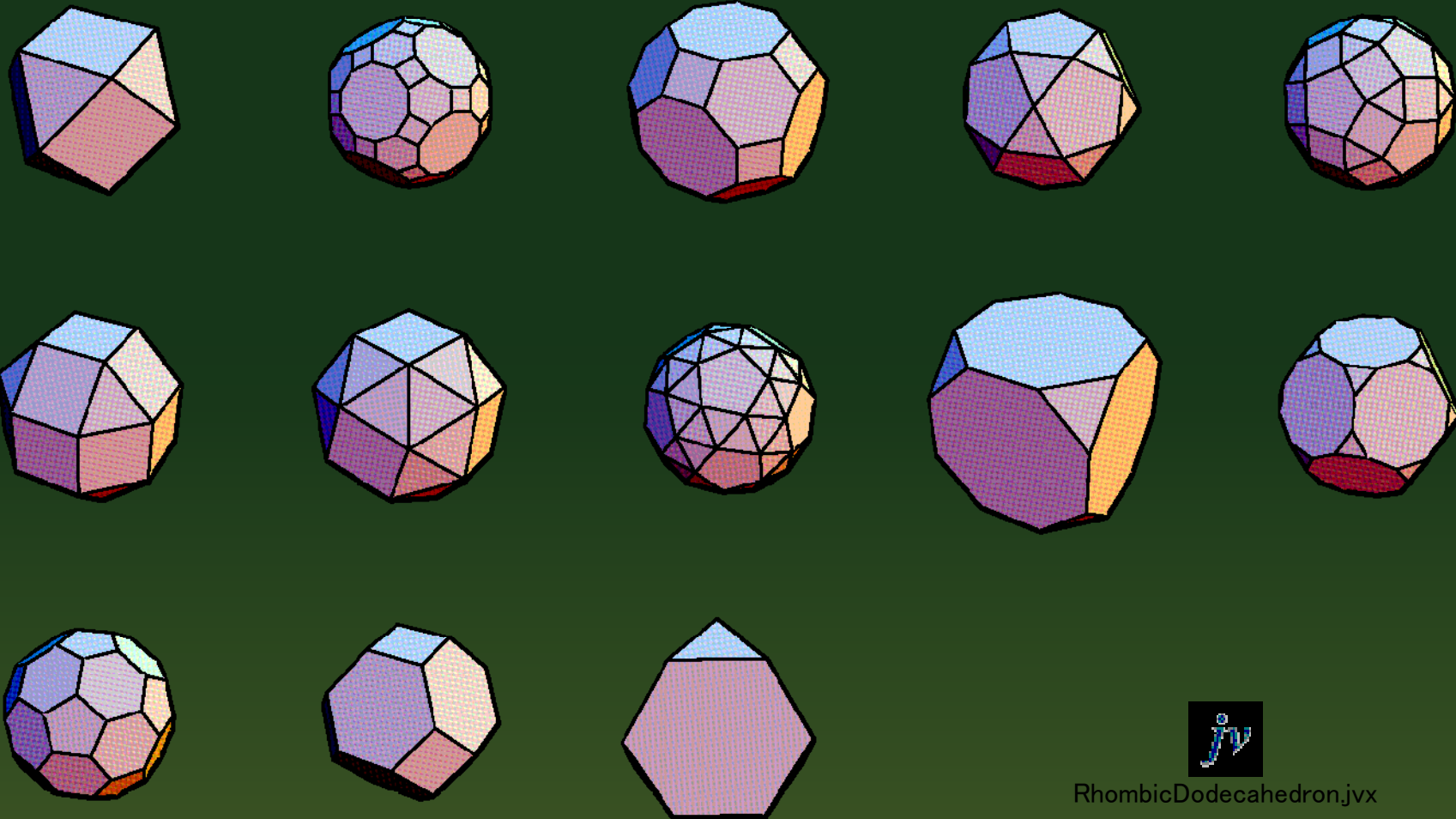
Dodecahedron.jvx



Icosahedron.jvx



Platon (427 BC-347 BC)



RhombicDodecahedron.jvx

Archimedean solid



Archimedes(287BC-212BC)

Euler

(1707—1783)



This is an article about Euler on the website of Professor Kanie, Mie University.

<http://www.com.mie-u.ac.jp/~kanie/tosm/humanind/jinmei.htm>

he was born in Basel, Switzerland, and died in St.Petersburg, Russia.

Euler is an apprentice of Johann Bernoulli, and was a member of Academy at Petersburg(27-41), Berlin(41-66), and Petersburg(66-83).

He became completely blind when he was 66, but even after that, he continued active research. He was an unsophisticated character, and had 13 children.

It is said that he did mathematics with babies in his arms and children playing near his feet. One day when he was calculating the orbit of Uranus, Euler called for his grandchildren, and then he took a cup of tea and in the middle of that, suddenly he told he was dying, and calmly “gave up his life and calculation”.

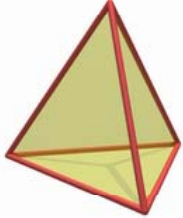
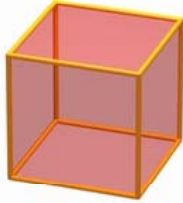

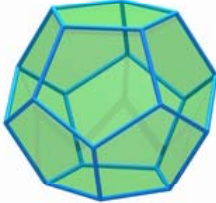

he was the most prolific mathematician through the history of mathematics. His academic achievements are found in analytics, algebra, number theory, probability, complex analysis, and variations. He studied problems such as how to connect all the bridges in Königsberg without lifting the pencil from the map, or the relation of the number of faces, edges, and vertices of polyhedrons (Euler characteristics), and now he is called “the father of graph theory and topology”. Euler class, Euler constant, Euler equation. Astronomy (computing solar eclipse, new lunar theory), Navigation (theory for rolling and pitching), fluid dynamics (motion of flying balloons) refractive optics (theory of lenses).

A French physicist Arago praised Euler as “an incarnation of analytics”, saying that “Euler could calculate with no apparent effort as men breathe or as eagles sustain themselves in the wind.” (English translation <http://www.hinduonnet.com/mag/2002/01/20/stories/2002012000210400.htm>)

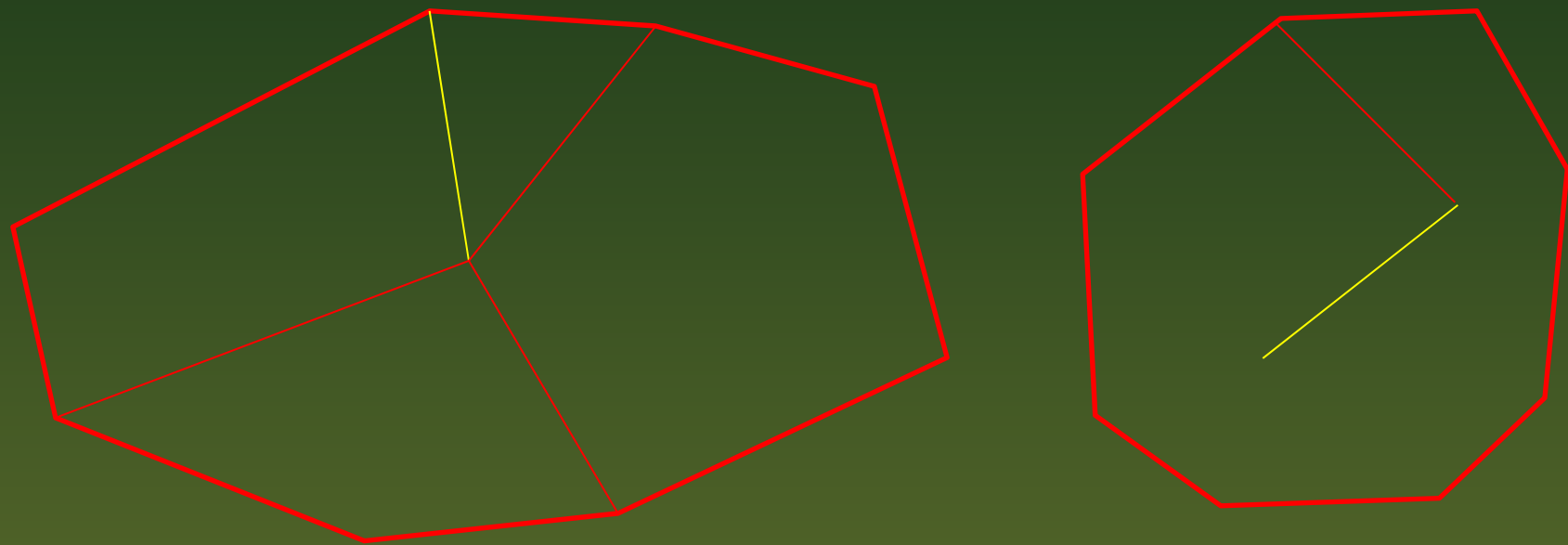
Euler's polyhedron theorem

For a polyhedron with the number of vertices, edges, and faces v , e , and f respectively,

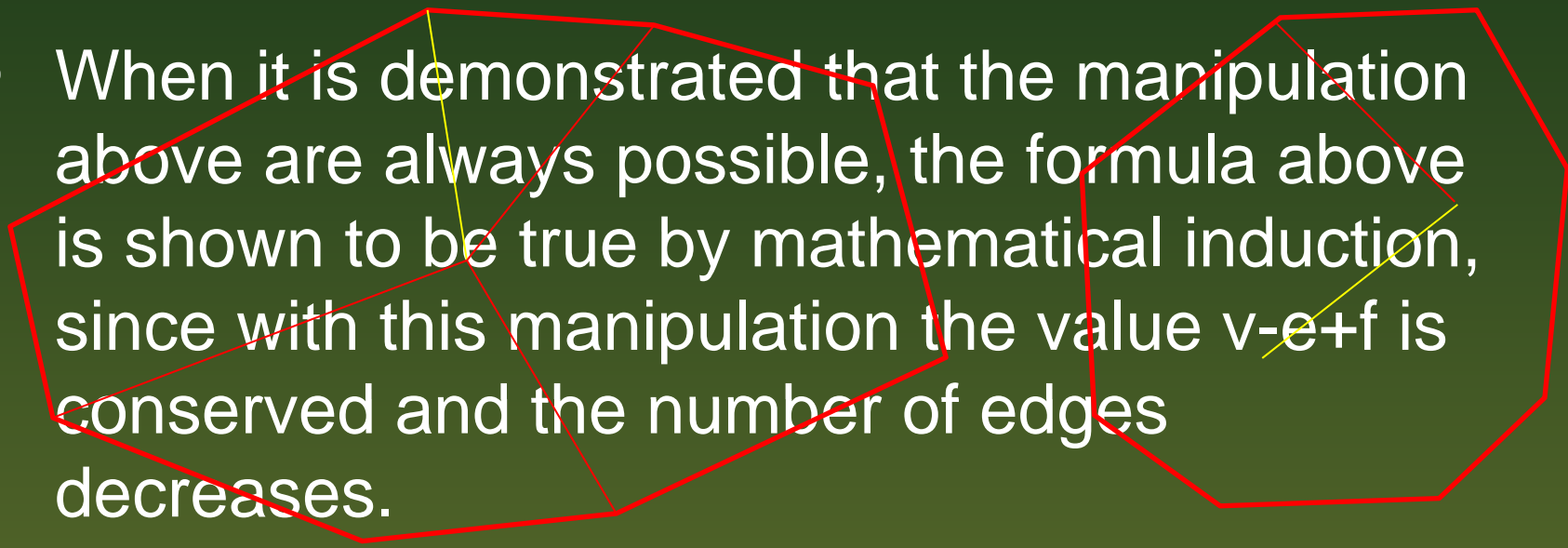
$$v - e + f = 2$$

					
Number of vertices v	4	8	6	20	12
Number of sides e	6	12	12	30	30
Number of faces f	4	6	8	12	20
$v-e+f$	2	2	2	2	2

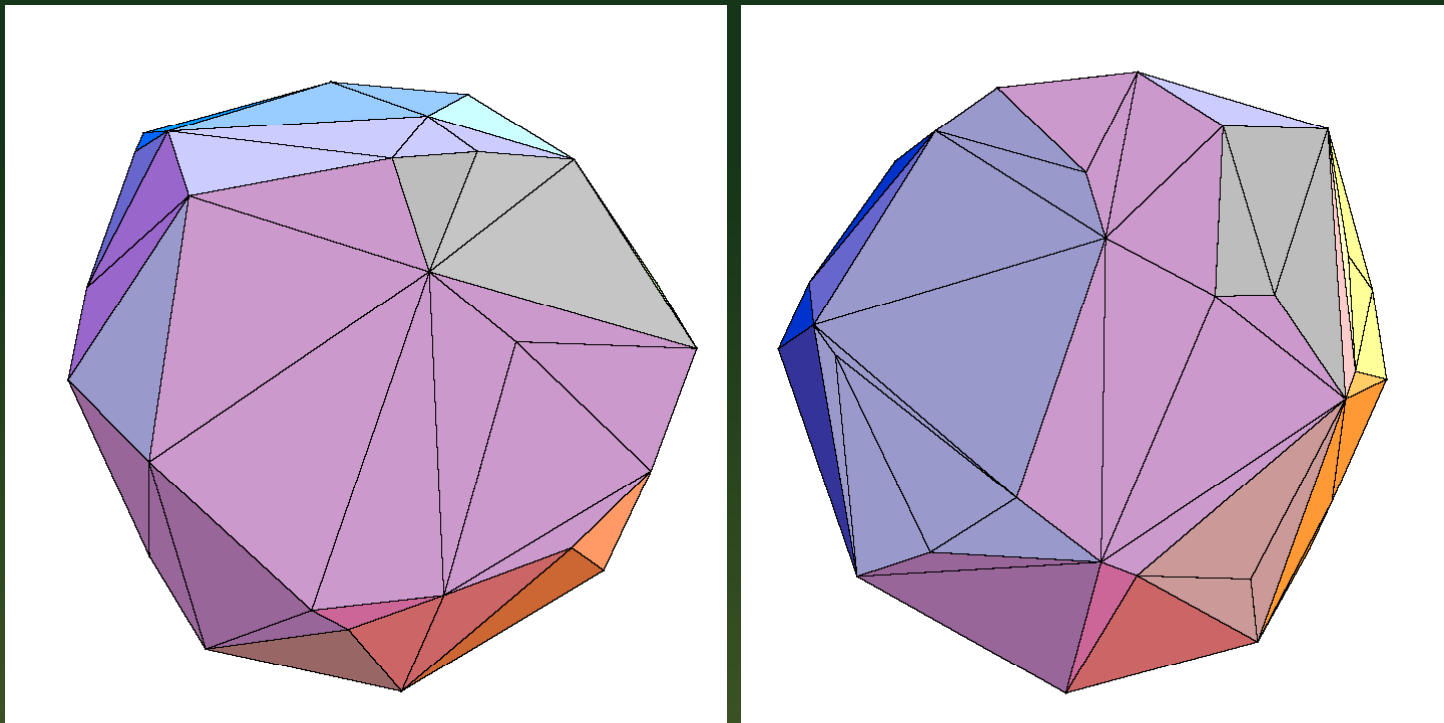
- on the separation of polygons on flat surfaces
- Polygons are defined as figures on flat surfaces whose sides are continuous.
- The number of edges and the number of vertices are equal.
- When a polygon on flat surfaces are divided into polygons by lines, we show that
$$v - e + f = 1$$
by mathematical induction with respect to the number of edges.



- before separating the polygon, the formula above are correct, since the number of faces is 1.
- Let us consider the following manipulation.
 1. Eliminate an edge so that neighboring two sides become connected.
 2. Eliminate edges that stick into the polygon.
- When it is demonstrated that the manipulation above are always possible, the formula above is shown to be true by mathematical induction, since with this manipulation the value $v-e+f$ is conserved and the number of edges decreases.



- When the number of faces are more than 2 there exist neighbouring faces, so it is possible to eliminate edges.
- if vertices and edges that stick into the polygon appears, we will eliminate vertices and edges from the head.
- Therefore, the formula $v-e+f=1$ is demonstrated when a polygon on the flat surface is divided into polygons with lines.



- The front and reverse side of a convex polyhedron K can be projected onto two polygons in the flat surface P_1 and P_2 .
- The intersection of front side and reverse side, C , is projected onto the boundary of P_1 and P_2 .
- Then

The number of the edges of K = the number of the edges of separation of P_1 + the number of the edges of separation of P_2 – the number of the edges of C

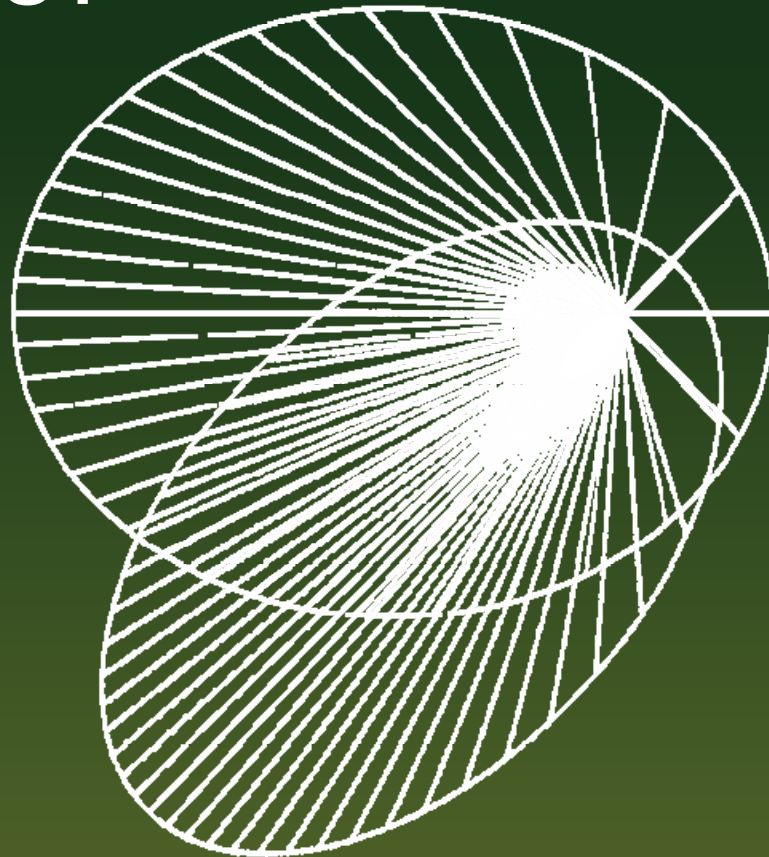
The number of the vertices of K = the number of the vertices of the separation of P_1 + the number of the vertices of the separation of P_2 – the number of the vertices of C

The number of the faces of K = the number of the faces of the separation of P_1 + the number of the faces of the separation of P_2

$$\begin{aligned}
& v(K) - e(K) + f(K) \\
&= v(P_1) + v(P_2) - v(C) \\
&\quad - (e(P_1) + e(P_2) - e(C)) \\
&\quad + f(P_1) + f(P_2) \\
&= 1 + 1 + e(C) - v(C) \\
&= 1 + 1 = 2
\end{aligned}$$

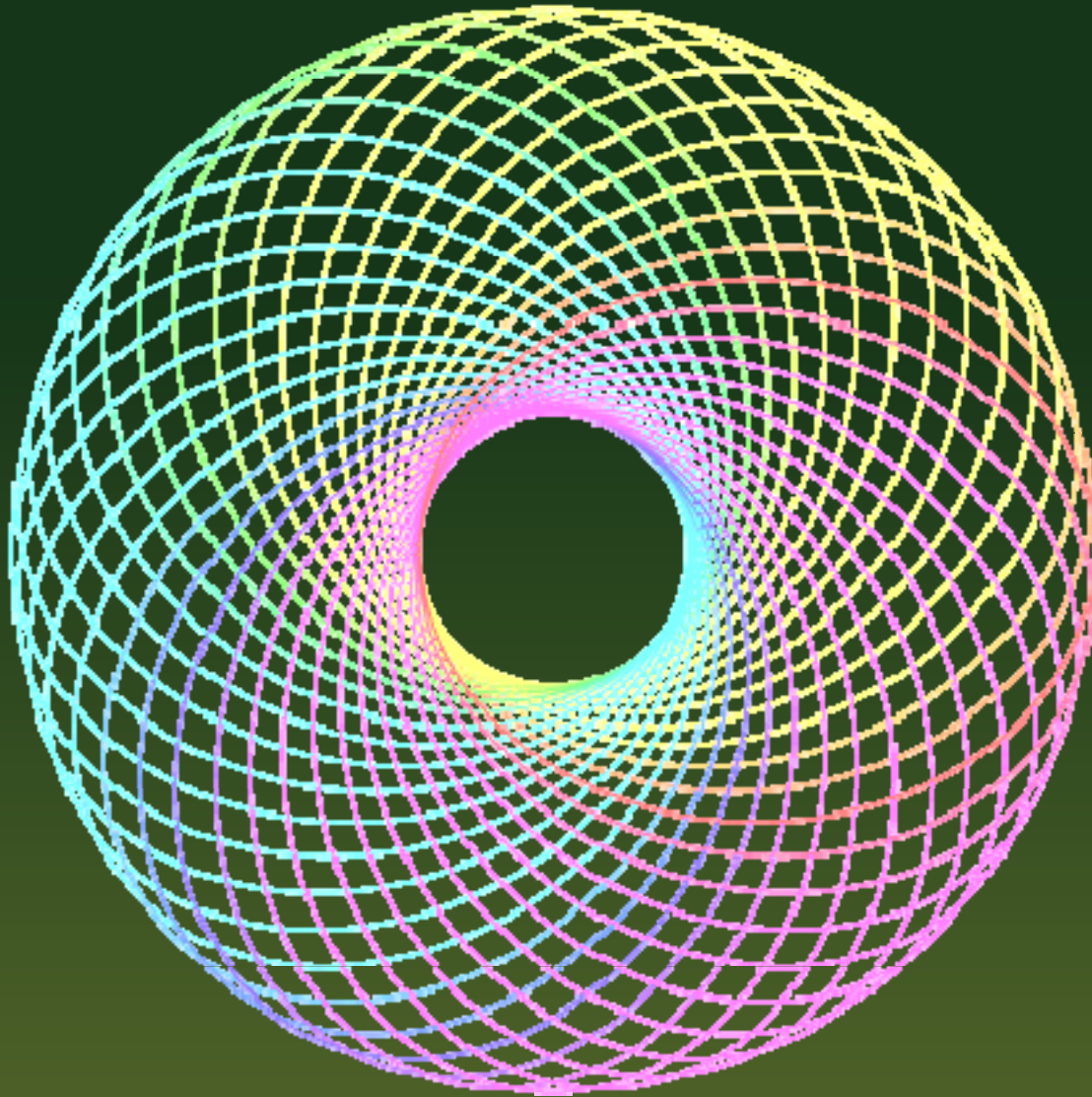
It is demonstrated that for a convex polyhedron with v vertices, e edges, and f faces, the relation $v - e + f = 2$ always holds.

What is the relation between Euler's polyhedra and planetary orbits?

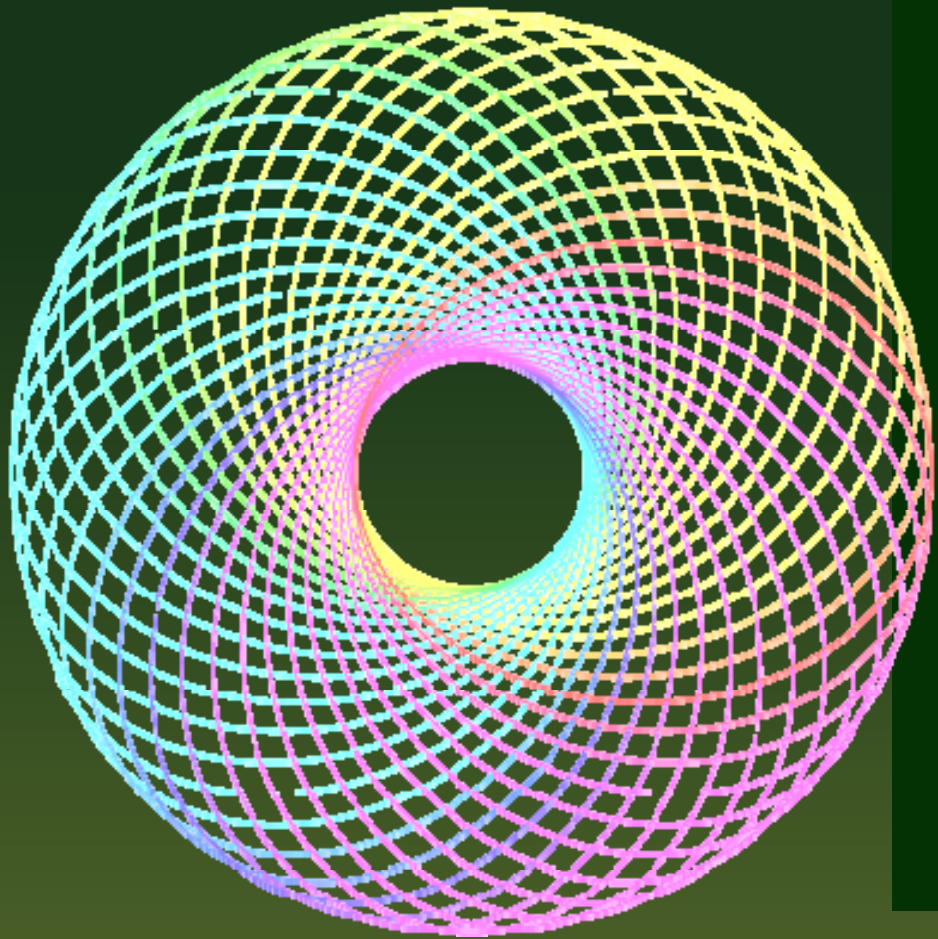


- When two stars are interacting only through gravitational force, the equation of motion is reduced to an equation of central force.
- We can take the coordinate system whose origin corresponds to the center of gravity of two stars, since this point is uniformly moving due to the law of action and reaction.
- The state of motion on a flat surface is represented by a point in four-dimensional space and is composed of the coordinate (2 dim) and velocity (2 dim) of the body.
- If you choose the initial coordinate and velocity, the equation of motion can be written down using the law of universal gravitation, and the motion will be determined.

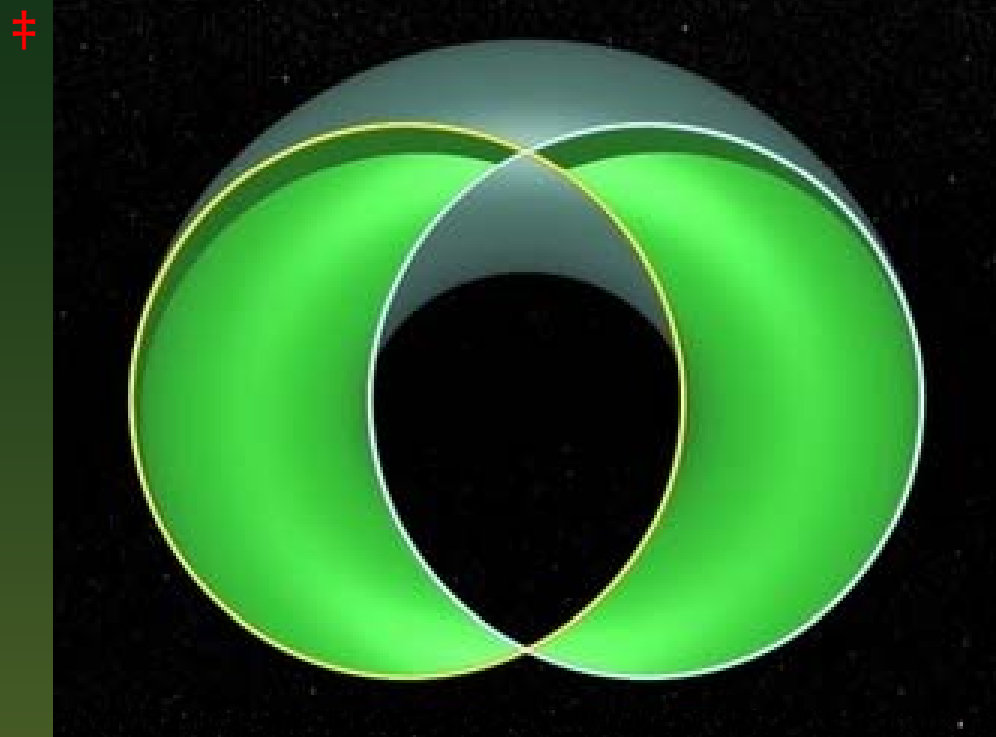
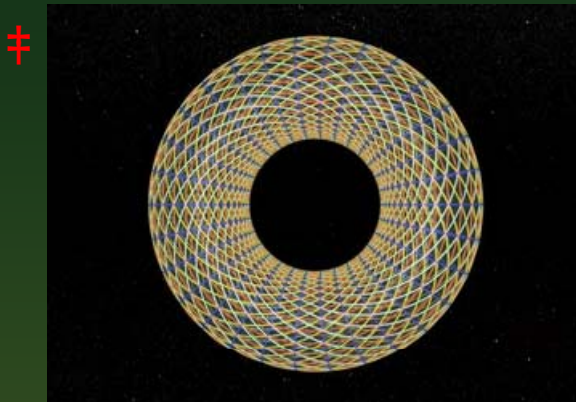
- In addition, we have two more relations, namely the conservation of angular momentum (the constant areal velocity with respect to the center of gravity) and the conservation of energy.
- We see a subspace in the four-dimensional space satisfying both relations. This is the space of state that describes the motion with constant momentum, angular momentum, and energy.
- If the solution is an ellipse, the space of state is a two-dimensional torus.



- Set the value of energy.
- Potential energy is determined by the distance from the center of gravity, and then the velocity is determined.
- since the areal velocity must be constant, there are two possible direction (one to the perihelion and one to the aphelion).



- These ellipses are related to the Villarceau circles on tori.
- For each point on a torus that can be obtained as rotated surface, there are four circles that passes through the point. Namely, longitudinal line, latitudinal line, and the two Villarceau circles.
- Orthogonally projecting this Villarceau circle, we obtain an ellipse that have focuses in common and have the same length of semi-minor and semi-major axes.

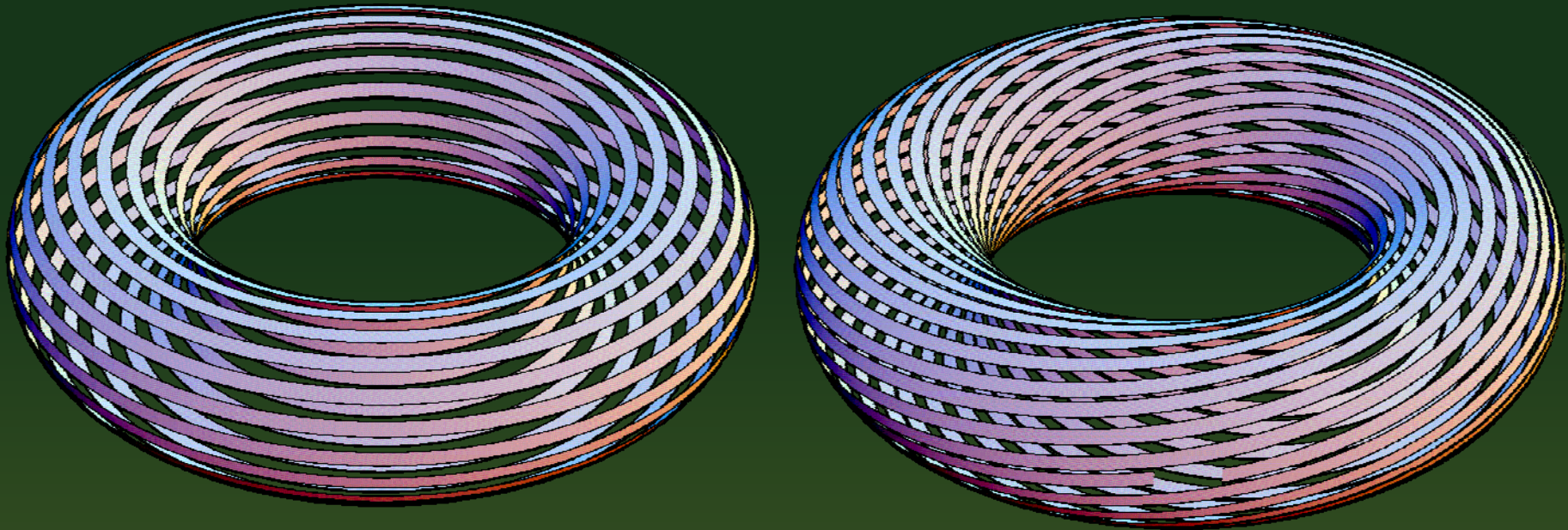


- the universal law of gravitation (inverse square law) is unnecessary for our discussion to reach the two-dimensional space of state.

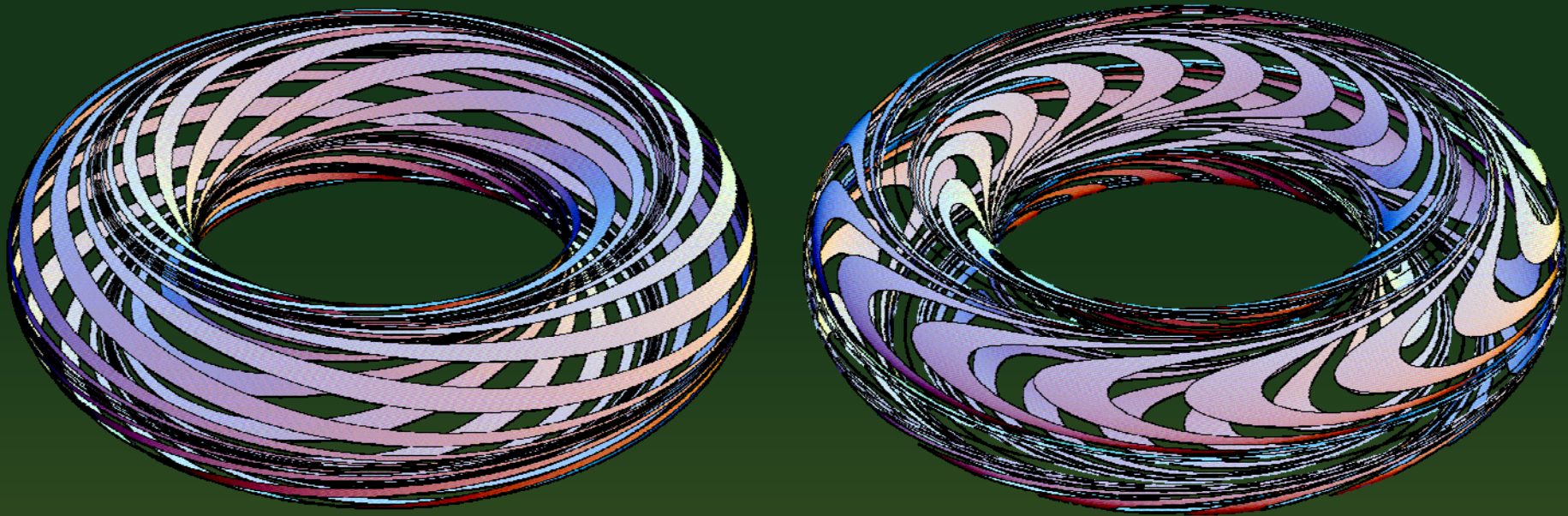
[BASIC code](#) for the inverse square law.

[BASIC code](#) slightly different from inverse square law.

- In general, let us consider a flow whose space of state is a torus.



- left figure indicates the case every orbit is an ellipse, and right figure indicates the case the inverse-square law is only approximately true.



- There exists such flows on a torus as shown above.
- For Hamiltonian systems, the area(volume) is conserved.
- So a flow like figure above does not appear as a solution for Hamiltonian systems.

Such two-dimensional spaces of state are, in general,
curved surfaces.



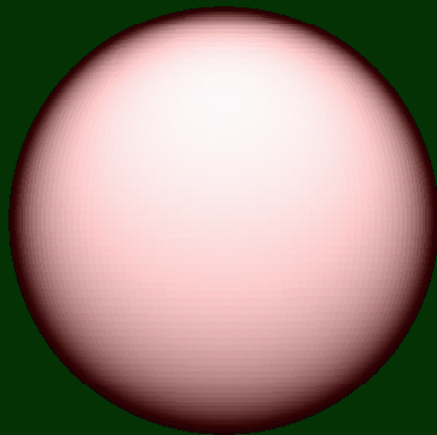
These are also bounded (compact) surfaces.

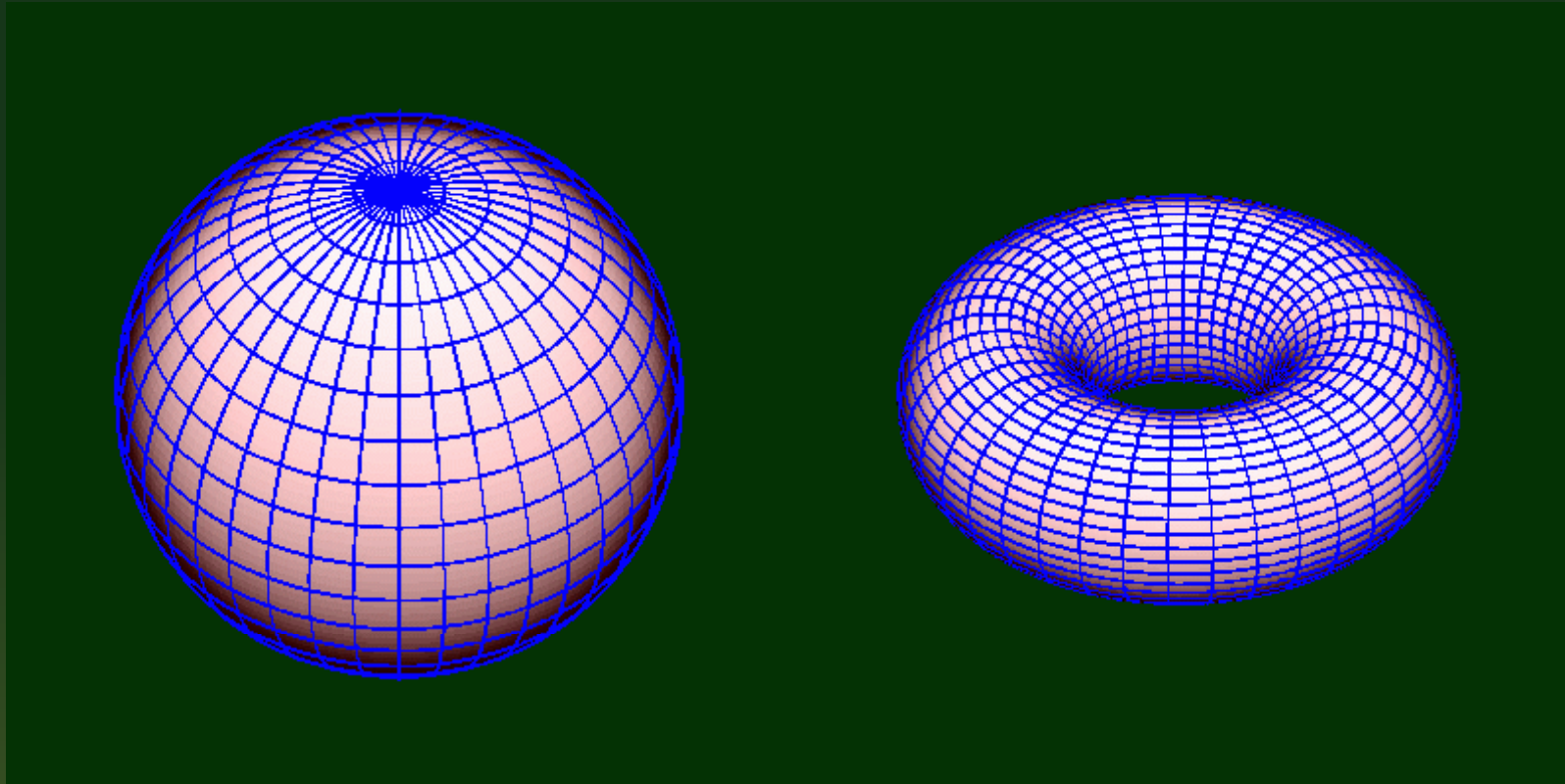


Flows on such surfaces should be different from flows on a torus.

In fact, there always appears stationary points (zero points of vector fields) on a sphere or on surfaces of genus more than 2.

Genus is the number of circles that can be embedded on the surface, keeping their complementary set connected.





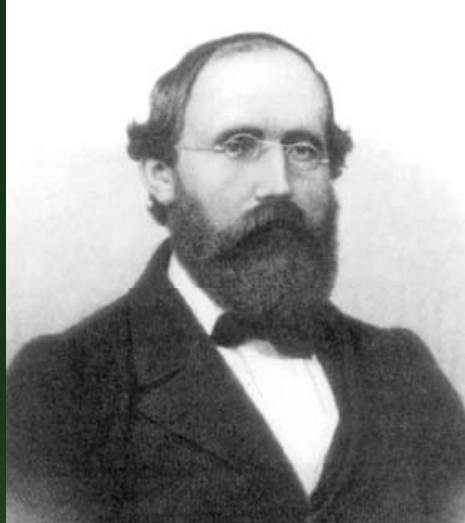
Typical flows on a surface and on a torus are shown above.

Though there are prior studies on curved surfaces before Archimedes, it is in the first half of 19th century that a huge breakthrough took place after Euler(1707-1783) .

Gauss
(1777—1855)



He discovered that curvature of a surface does not depend on how it is placed in space, and that it is determined by the length along the surface.



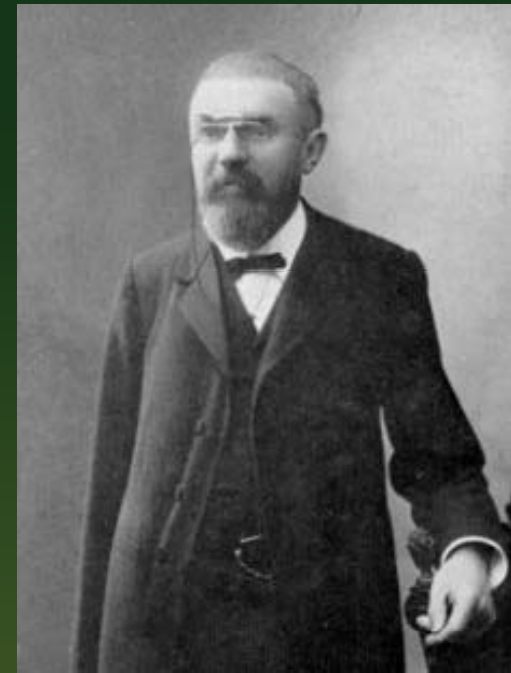
Riemann (1826 — 1866)

He formulated
spaces with metric.

Theory of Riemann surfaces, Riemann mapping theorem



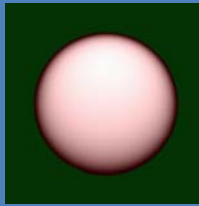

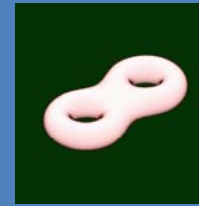
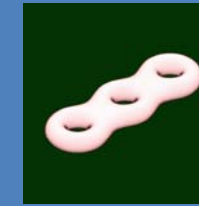
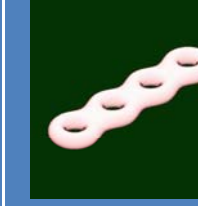
Klein (1849 — 1925)



Poincaré (1854 — 1912)

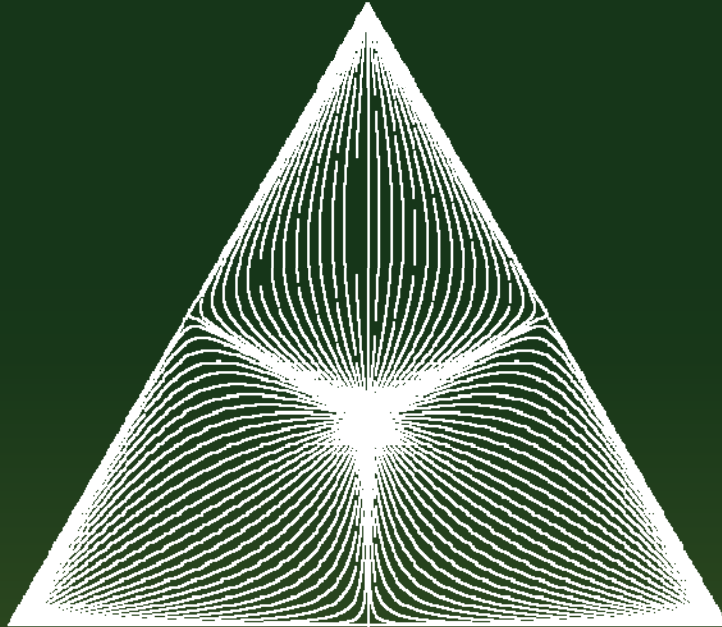
1760	L.Euler	Principal curvature of curved surfaces, Euler's formula
1760	J.L.Lagrange	equation for minimal surfaces
1784	G.Monge	curvature lines, definition of radius of curvature
1784	A.M.Legendre	Legendre transformation
1813	P.C.F.Dupin	concept of curves in space, definition of tangent spaces to curved surfaces
1827	C.F.Gauss	spherical representation, total curvature, Gauss' theorem, Gauss' theorem on spherical triangle
1836	C.G.J.Jacobi	jacobi field, conjugate points
1847	J.F.Serret	Frenet-Serret formula for spatial curves
1848	Bonnet	general Gauss-Bonnet theorem
1854	B.Riemann	on the assumption that lies in the fundamentals of geometry- Riemannian geometry –
1867	Bonnet	proof of fundamental theorem of curved surfaces using Gauss-Codazzi-Mainardi equation
1866	Darboux	thesis: on trifold orthogonal systems
1872	F.Klein	the Erlangen's catalog – compared group theoretic study of geometry
1898	J.S.Hadamard	topological type of complete simply-connected surfaces with negative curvature
1899	D. Hilbert	impossibility theorem of complete simply-connected surfaces

You can separate a curved surface into many triangles, so it is transformed to the surface of a polyhedron.
 the Euler characteristic = vertices – edges + faces is independent of how you separate the figure.
 The respective values are shown below:

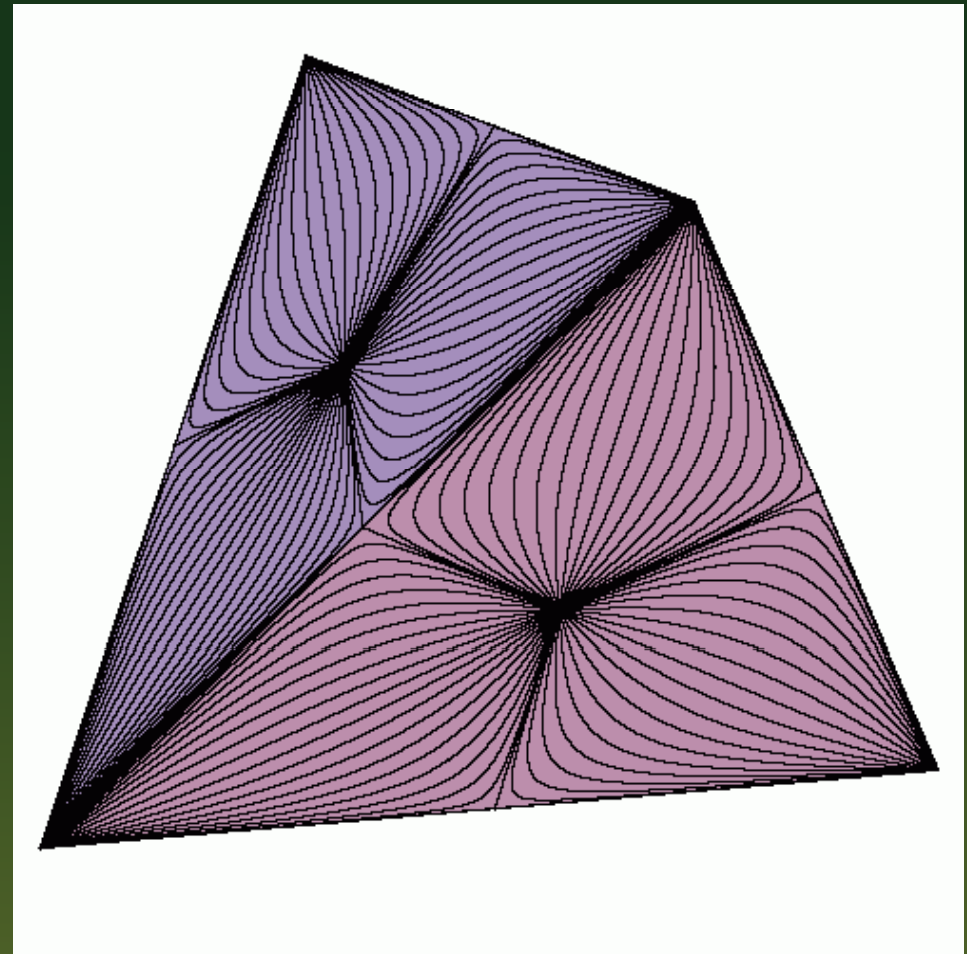
Orientable closed surface					
Euler characteristic	2	0	-2	-4	-6
genus	0	1	2	3	4

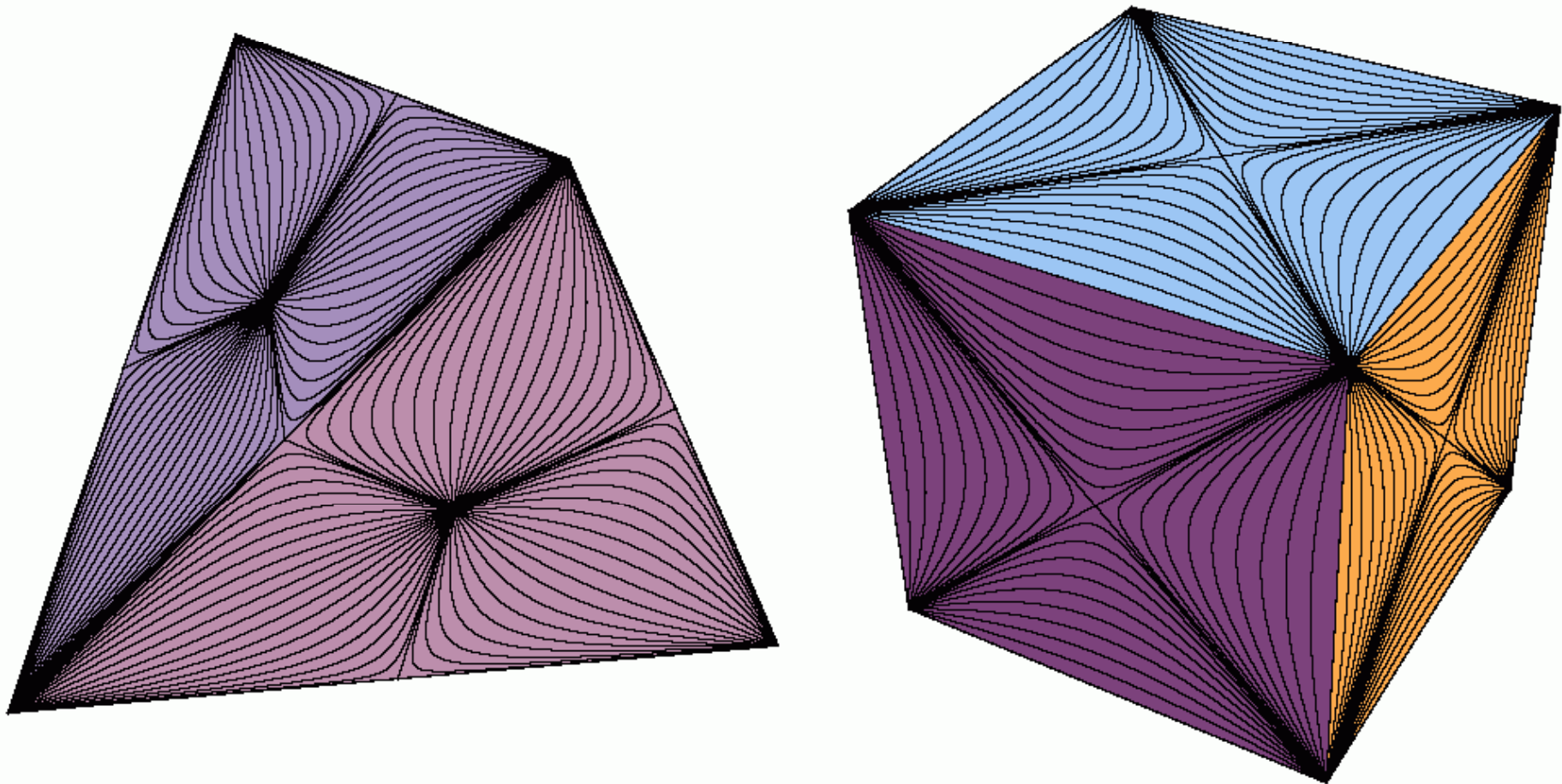
There always exists a stationary point (zero point of a vector field) on a sphere or on a curved surface of genus more than two, and the reason for this is that Euler characteristic of these surfaces are not zero.
 So why does this occur?

On each triangle with which you separate a polyhedron, you can draw a pattern shown below:

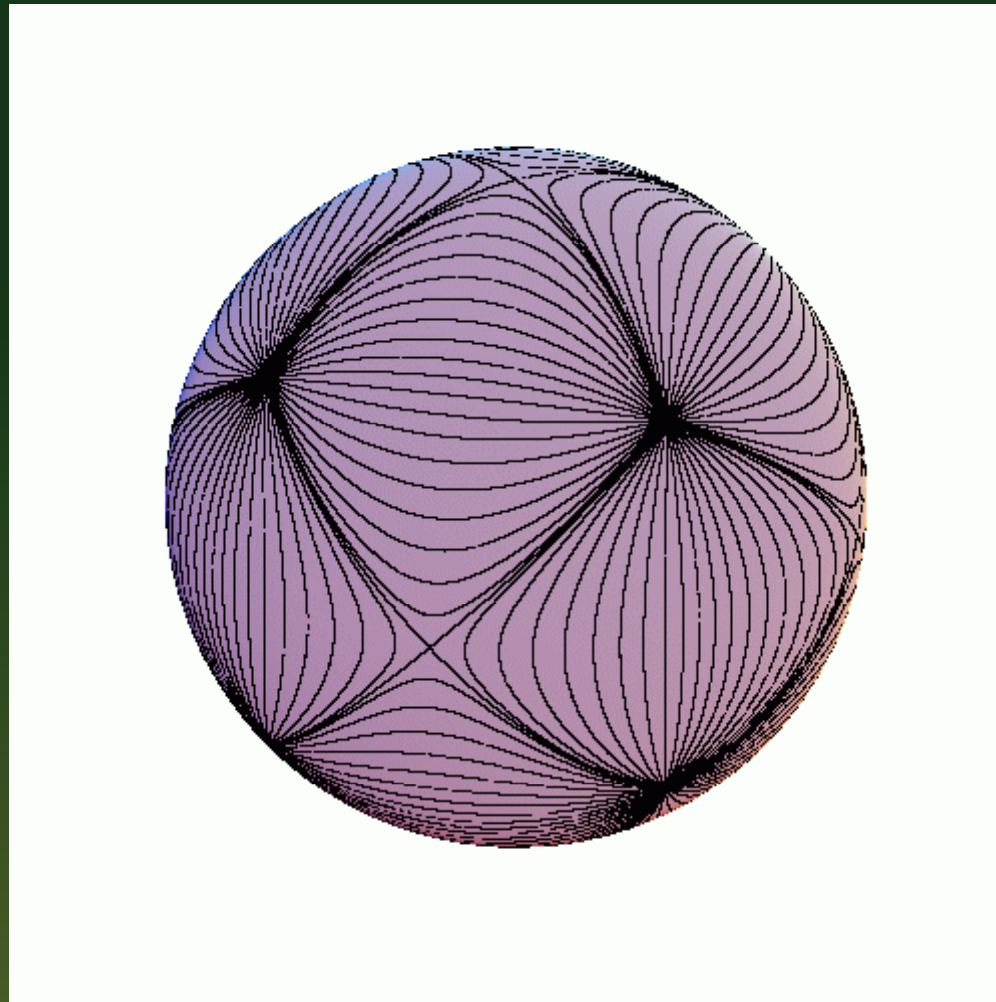


Then a polyhedron will be like:

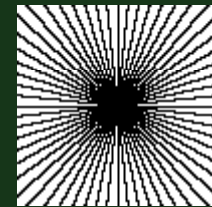
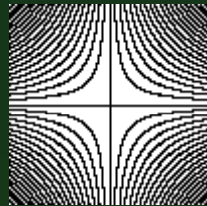




You can draw the pattern on a sphere:

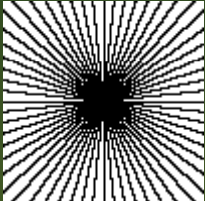
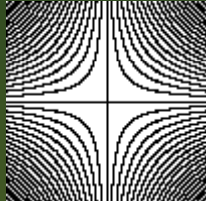


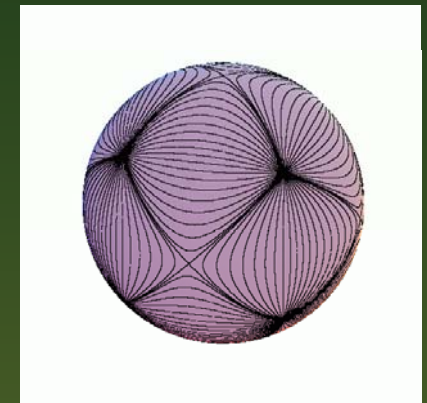
Vertices v and faces f of Euler's polyhedron theorem correspond to
 and edges e to



Therefore for a pattern on a sphere, statement of Euler's polyhedron theorem is rewritten as follows:

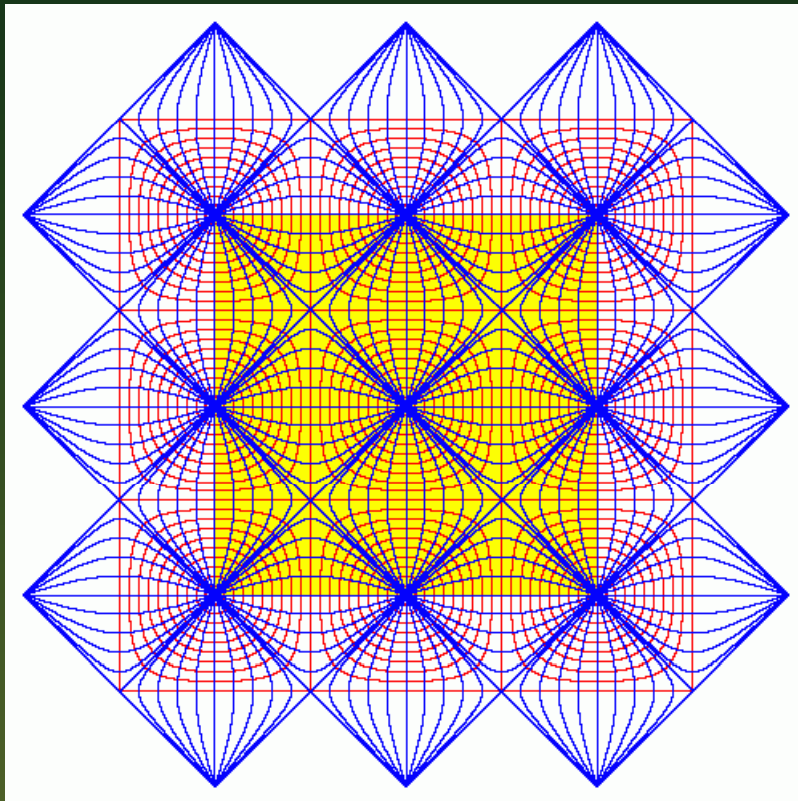
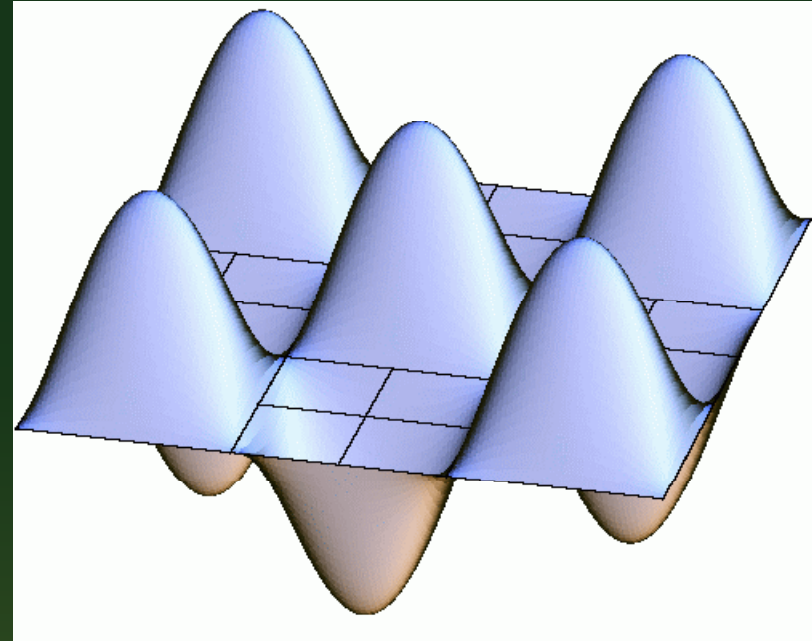
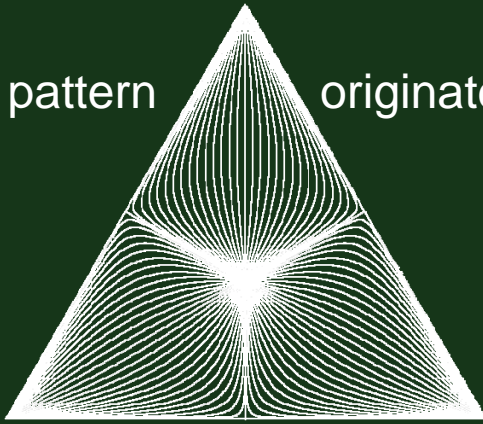
Poincaré-Hopf theorem

Number of  - Number of  = 2



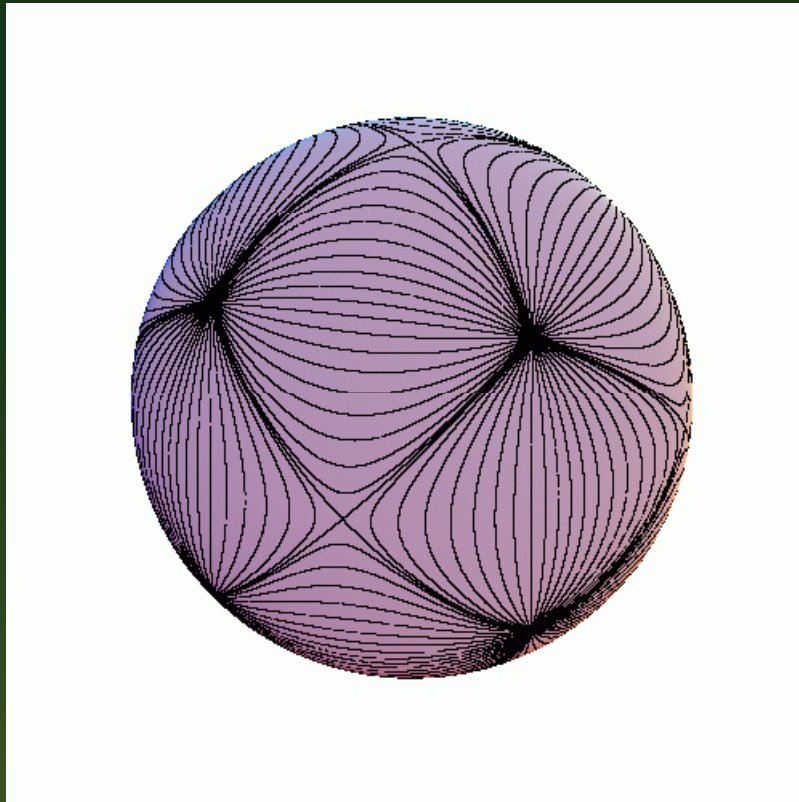
for general surfaces, right hand side is the Euler characteristic.

This pattern originates from the graph of the function $\cos x \cos y$

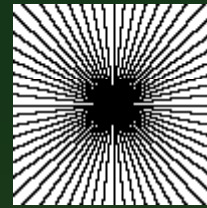


For each contour (red), there are orthogonal maximal gradient lines (blue)

You can fill the triangle with the pattern of maximal gradient lines.

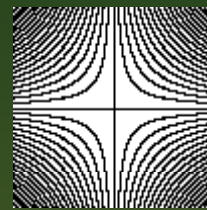


if a flow like this
pattern exists,



corresponds to

maxima or minima,



corresponds to

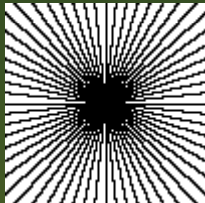
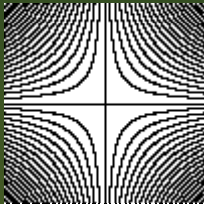
saddle points.

Euler's polyhedron theorem is demonstrated with such manipulations as “eliminate edges so that the number of faces decreases”, and “contract edges and vertices”

This operation, for the pattern of maximal gradient lines,

[Corresponds to the elimination of a bump.](#)

By deforming a surface,

The pair  and  and

can be eliminated.

Poincaré-Hopf theorem

When a flow on a compact surface has stationary point





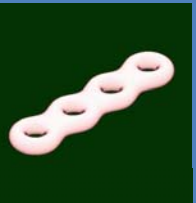
of type  and type .

Number of  - Number of  = the Euler

characteristic of the surface

If there exists a flow without any stationary points on a compact surface, the Euler characteristic of such a surface is 0.

if there exists a flow without any stationary points on a compact surface, the surface is homeomorphic to (in other words, the surface is the same figure as) a torus or a Klein bottle.

Orientable closed surfaces					
Euler Characteristic	2	0	-2	-4	-6
genus	0	1	2	3	4

References

DIMENSIONS

<http://www.dimensions-math.org/>