Global Focus on Knowledge　～Creating Mathematics～

Lecture four

Co-evolution of Numbers and Geometry

Creating words, Originating worlds

2009.10.29

Establishment of arithmetic geometry(late 20th century)

- Research on the figures defined by equations with integer coefficient,

  like $X^n + Y^n = Z^n$

  or

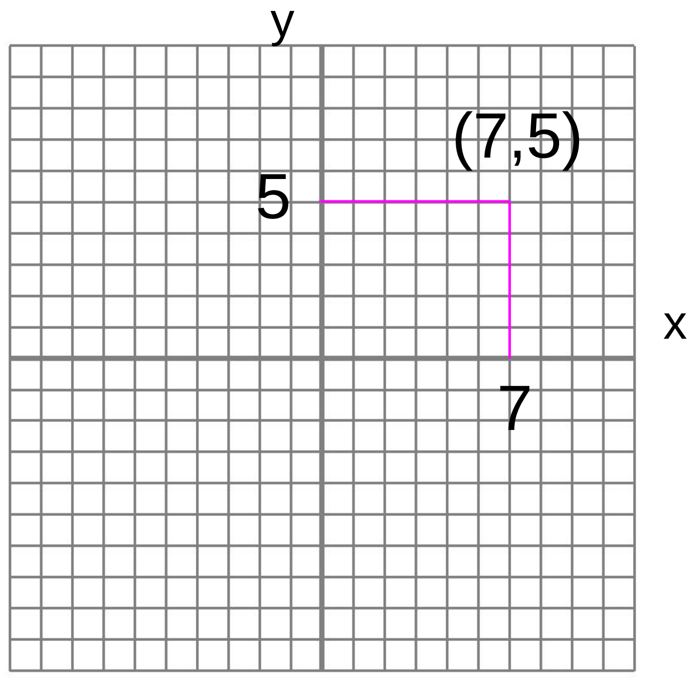
- Research on the properties of integers using such figures

The relation between number and figure-coordinates

- Numbers and figures are the two sides of the same thing

Descartes 1637

『Discourse on the method』



Descartes (1596.3.31 - 1650.2.11)

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High-dimensional spaces

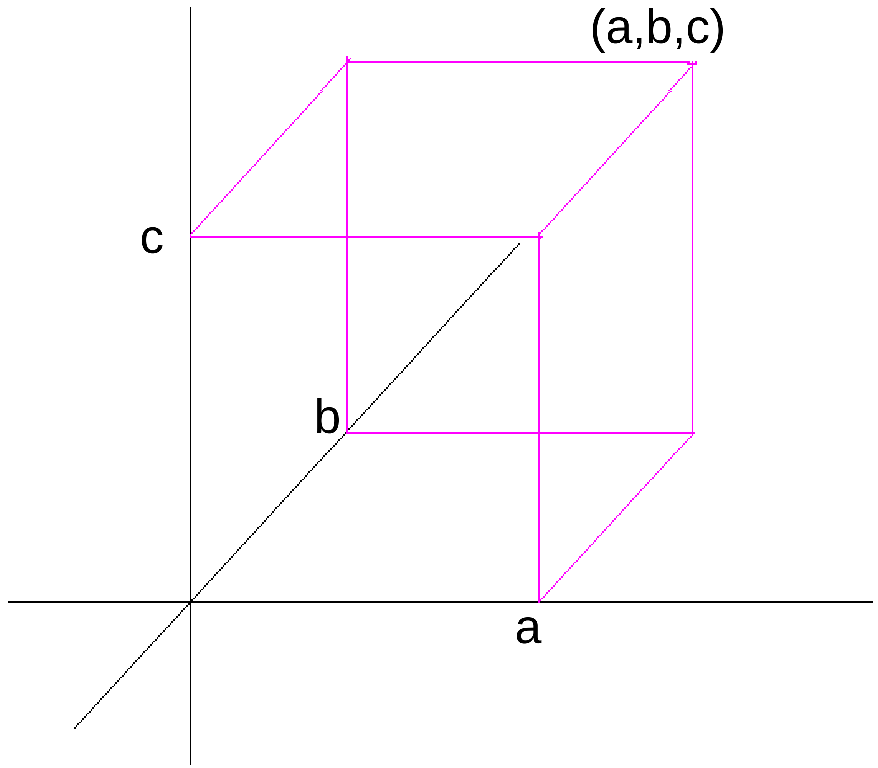
- Also in the case of n>3,

  A set of n numbers

  $(x_1, x_2, \ldots, x_n)$

  indicates

A point on a n-dimensional space



High dimensional space ="the world of linear algebra"

- invisible
- Based on simultaneous linear equation

reduced to linear algebra

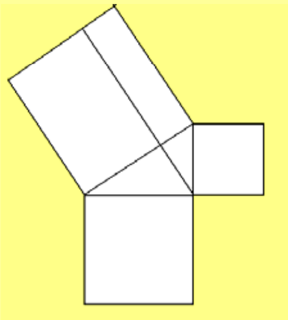
‖

mathematically well known

Paradigm shift in geometry

Figures

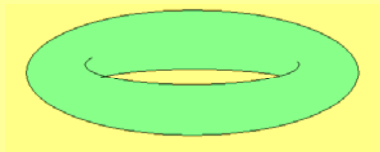
in planes or spaces



research subject

Space

itself



Riemann 1854

“on the hypothesis which lies in the

fundamental of geometry”

Elliptic curves and Fermat's last theorem

- In the case of $n = 3, 4$

  Demonstrated by studying the rational number solution of the equations which determine certain elliptic curves

  $$y^2 = x^3 - x \qquad \text{(Fermat)}$$

  $$y^3 = x^3 - 1 \qquad \text{(Euler)}$$

- In the case that n is a prime number and larger than or equal to 5

  Demonstrated by showing that the equation that determines an elliptic curve $y^2 = x (x-a^n) (x-c^n)$ does not exist

Riemann (1826.9.17 - 1866.7.20)

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Paradigm shift in physics

Quantum mechanics

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Super-Kamiokande

elementary particles

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Paradigm shift in biology

molecular biology

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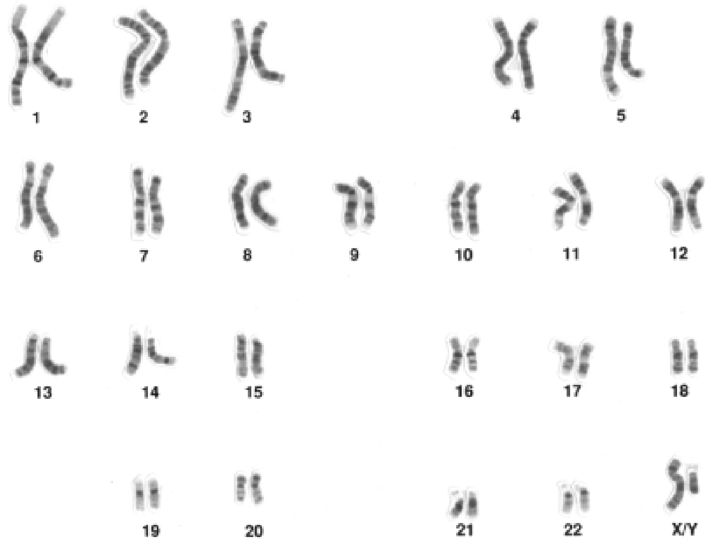


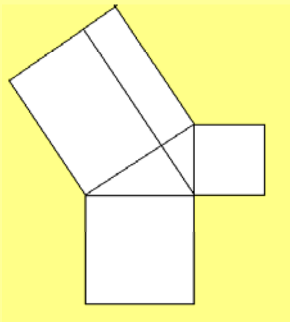
genes

Paradigm shift in geometry

modern mathematics

Figures

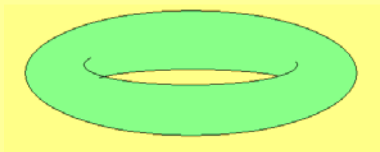
in planes or spaces



→

Space

itself



manifolds

Where the paradigm shift in geometry come from?

- Riemann surfaces
- Gauss' theory on curved surfaces
- Non-Euclidean geometries
- Physics

…………

Multi-valued function
(somewhat contradictory name)

- function : the rule that relates a point (=variable) to one value
- The square root of z

if z is a real number

you can take the positive one

if z is a complex number

impossible to specify

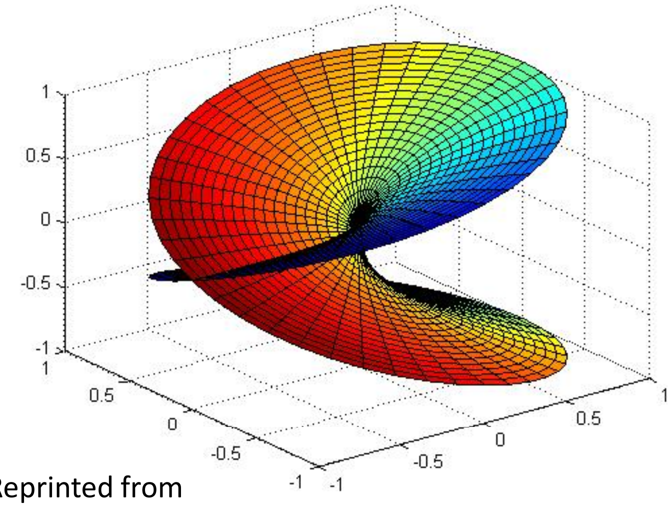
which one you should take

Riemann surfaces

- You can consider the complex plane as the superposition of two sheets

on the one sheet $\sqrt{z}$ on the other $-\sqrt{z}$

- The two sheets are actually connected



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Establishment of the concept of function

- Consider the "double" covering of complex plane:

  "multi-valued" function(somewhat contradictory name) now consistent

- "double" covering is sticking out of complex plane = Riemann surface

Gauss' theory on curved surfaces

he discovered a geometric property of curved surfaces,

and the property is unrelated to external 3-dimensional spaces

Studying on the space?

- The description of the object

  figures in unknown spaces

  how to make up?

- Method for the study

  what does it mean to understand spaces?

Set as a term in mathematics

- By regarding a set as a point or an element of some other set,

  we can create a new object

The emergence of “modern mathematics”

=

A “math” based on the concept of “set and topology”

Mathematics as a language in natural sciences

- the universe ...is written in the language of mathematics

  Galileo Galilei “the Assayer”

- Modern mathematics is written by the words of set theory

Paradigm shift

The common sense :

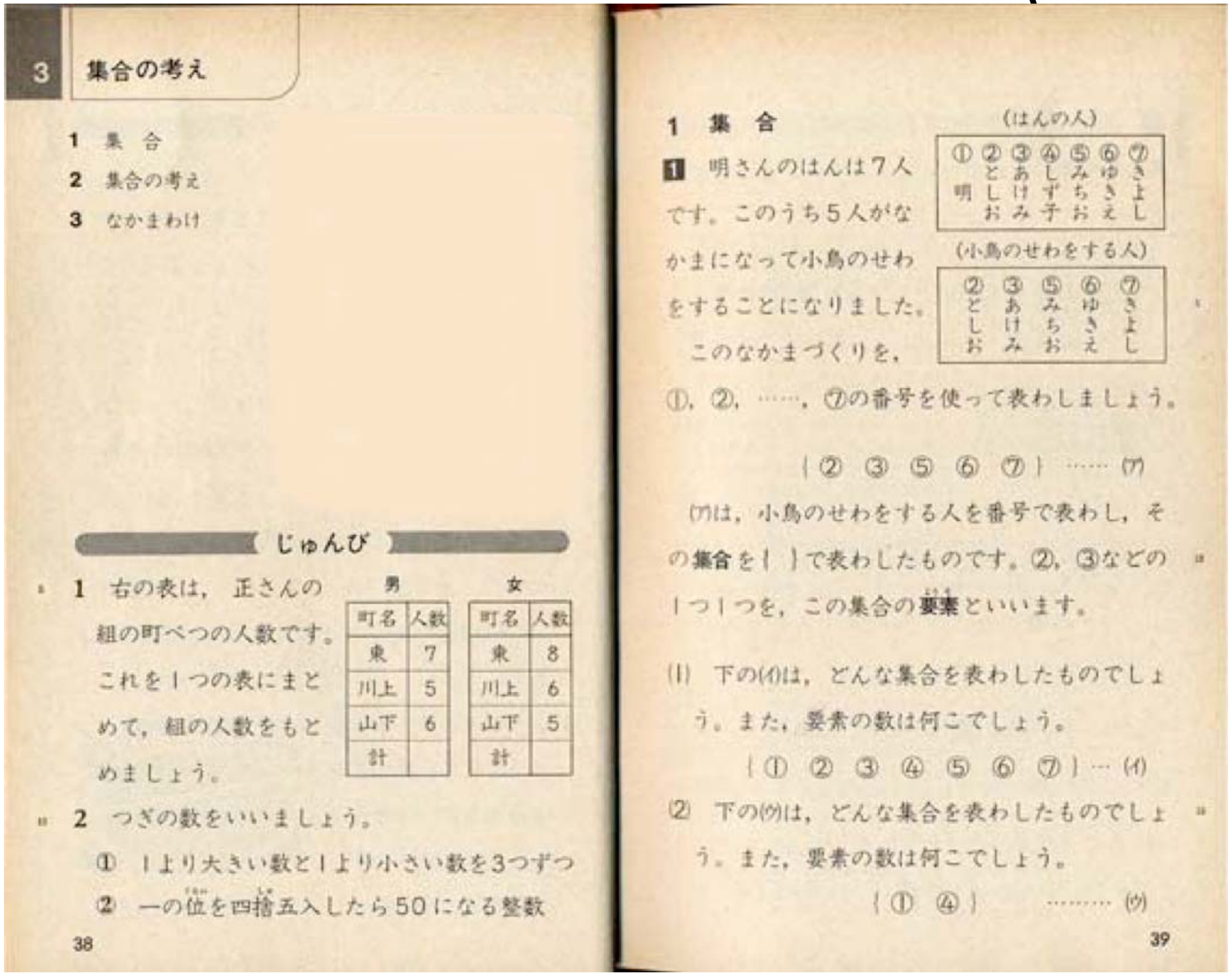
changes with times

How times change :

when a new generation appears, new ways of thinking become popular

“Modernization” in math education(1971-79)

‡



3 集合の考え

じゅんび

1 右の表は，正さんの組の町べつの人数です。これを１つの表にまとめて，組の人数をもとめましょう。

男

町名	人数
東	7
川上	5
山下	6
計	

女

町名	人数
東	8
川上	6
山下	5
計	

2 つぎの数をいいましょう。

① １より大きい数と１より小さい数を３つずつ

② 一の位を四捨五入したら50になる整数

1 集　合

1 明さんのはんは７人です。このうち５人がなかまになって小鳥のせわをすることになりました。

このなかまづくりを，①，②，……，⑦の番号を使って表わしましょう。

(はんの人)

①	②	③	④	⑤	⑥	⑦
明	としお	あけみ	しず子	みちお	ゆきえ	きよし

(小鳥のせわをする人)

②	③	⑤	⑥	⑦
としお	あけみ	みちお	ゆきえ	きよし

{② ③ ⑤ ⑥ ⑦} …… (ア)

(ア)は，小鳥のせわをする人を番号で表わし，その**集合**を{ }で表わしたものです。②，③などの１つ１つを，この集合の**要素**といいます。

(1) 下の(イ)は，どんな集合を表わしたものでしょう。また，要素の数は何こでしょう。

{① ② ③ ④ ⑤ ⑥ ⑦} … (イ)

(2) 下の(ウ)は，どんな集合を表わしたものでしょう。また，要素の数は何こでしょう。

{① ④} ……… (ウ)

Dedekind cut (1858.11.24)

- A method that treats real number regarding rational numbers as elements of set
- A real number x is a set of rational numbers

$$\{ r \mid r \text{ is rational and } r < x \}$$

- Hence, a real number equation $x = y$ means

$$r < x \iff r < y$$

for any given rational number r

manifolds = geometrical objects

patchwork of already known spaces

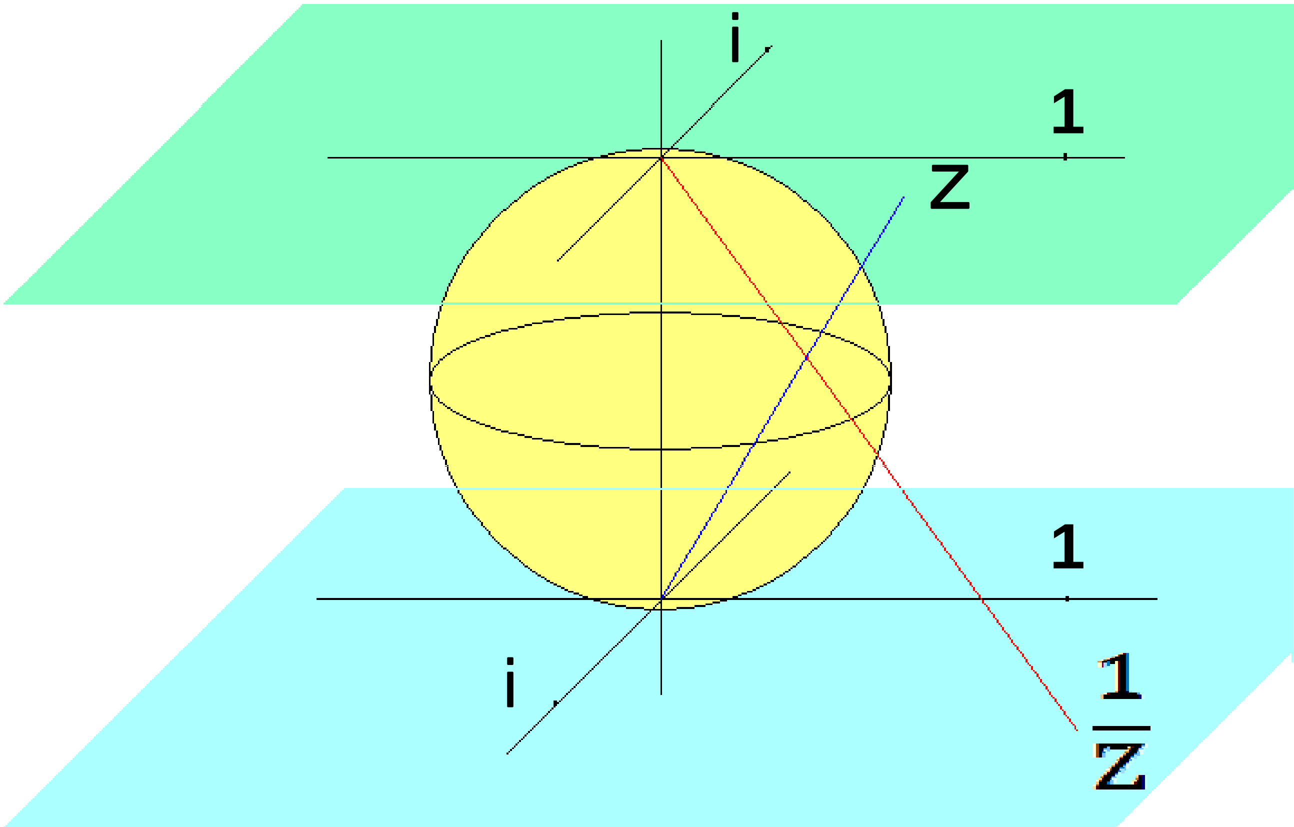
create a new space

manifolds = geometrical objects

- A space that cannot be embedded in an n-dimensional space $\mathbf{R}^n$ on which the coordinates are already defined
- The Einstein's universe :

  fundamentals of general relativity(1916)

Riemann sphere



invariant = methods in geometry

dimension, for example,

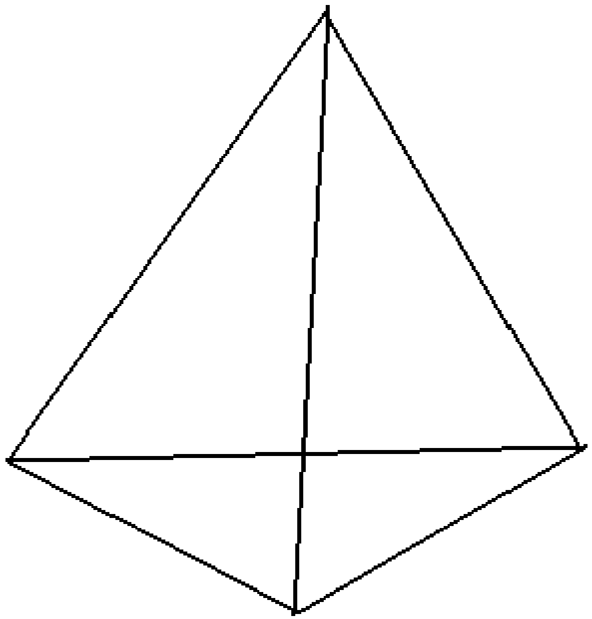
independent of the construction or the expression,

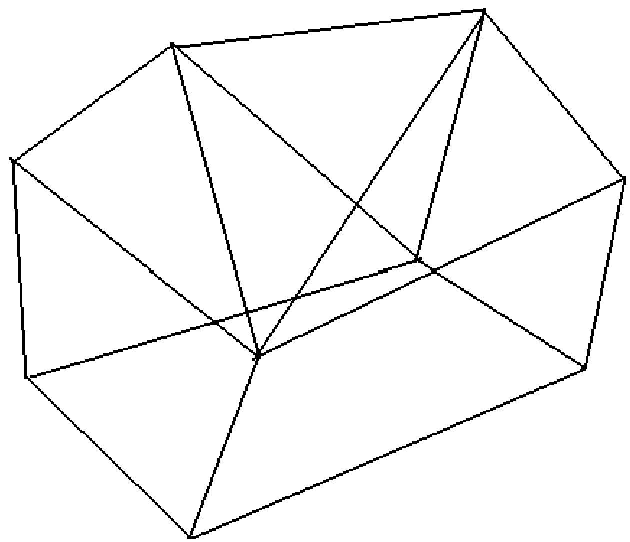
They are the same value for the same space

distinguish spaces using invariants

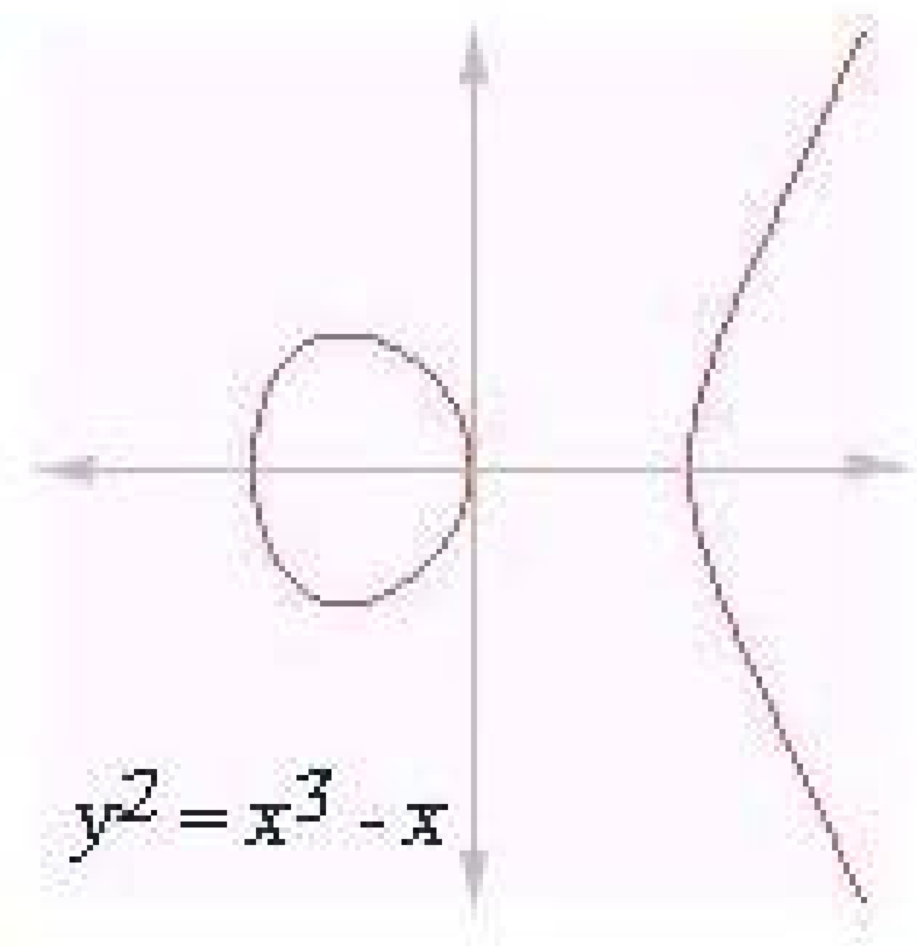
Euler characteristic for polyhedra

- (number of vertices)-(number of edges)+(number of faces)
- 4 – 6 + 4 = 2 9 – 16 + 9 = 2





Elliptic curves



x , y

real

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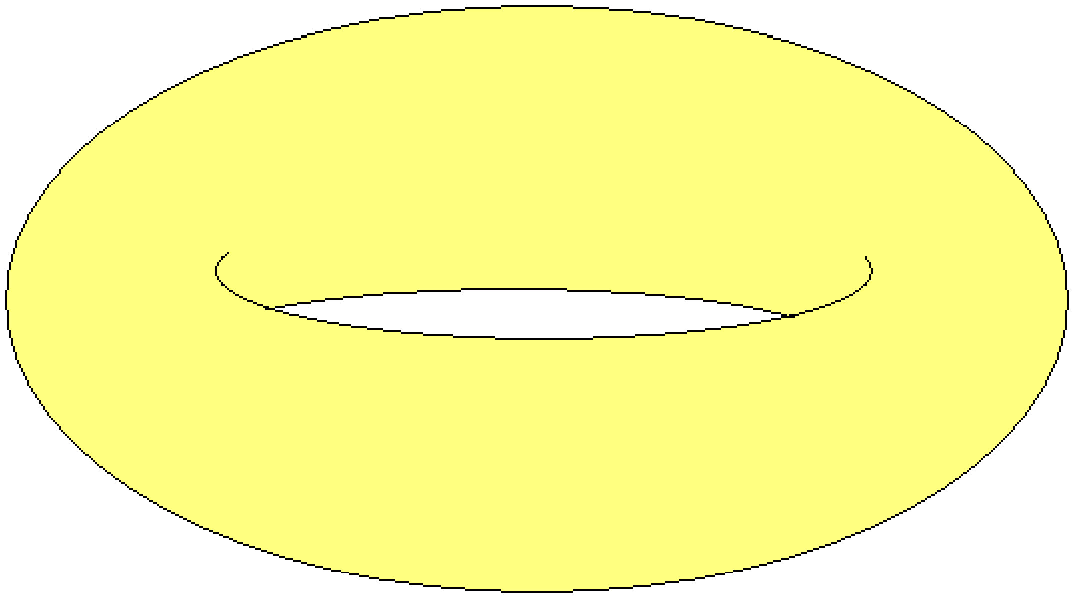
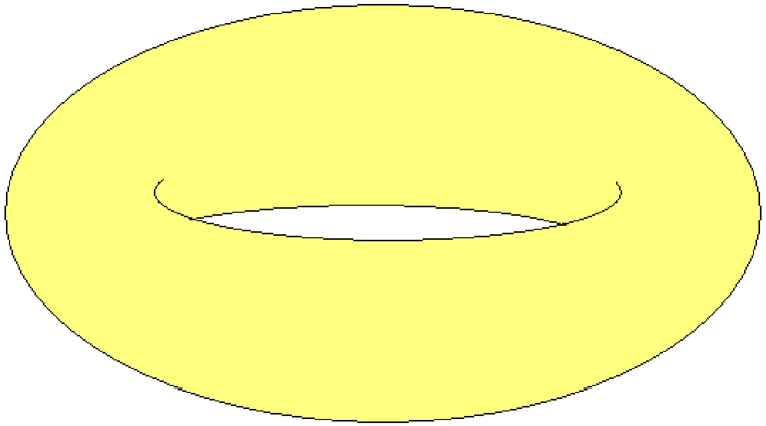
Elliptic curves as Riemann surfaces

- $y^2 = x^3 - x$ x , y complex number



Elliptic curves



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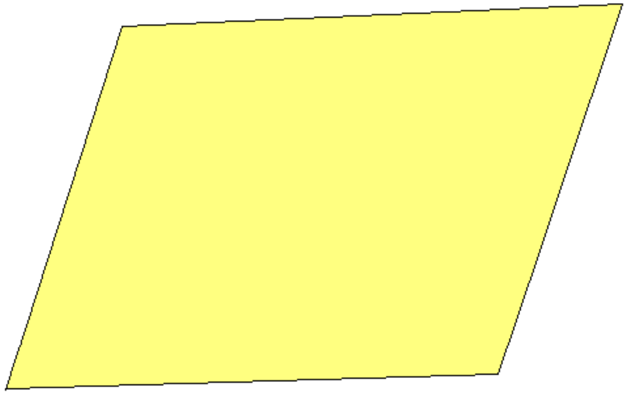
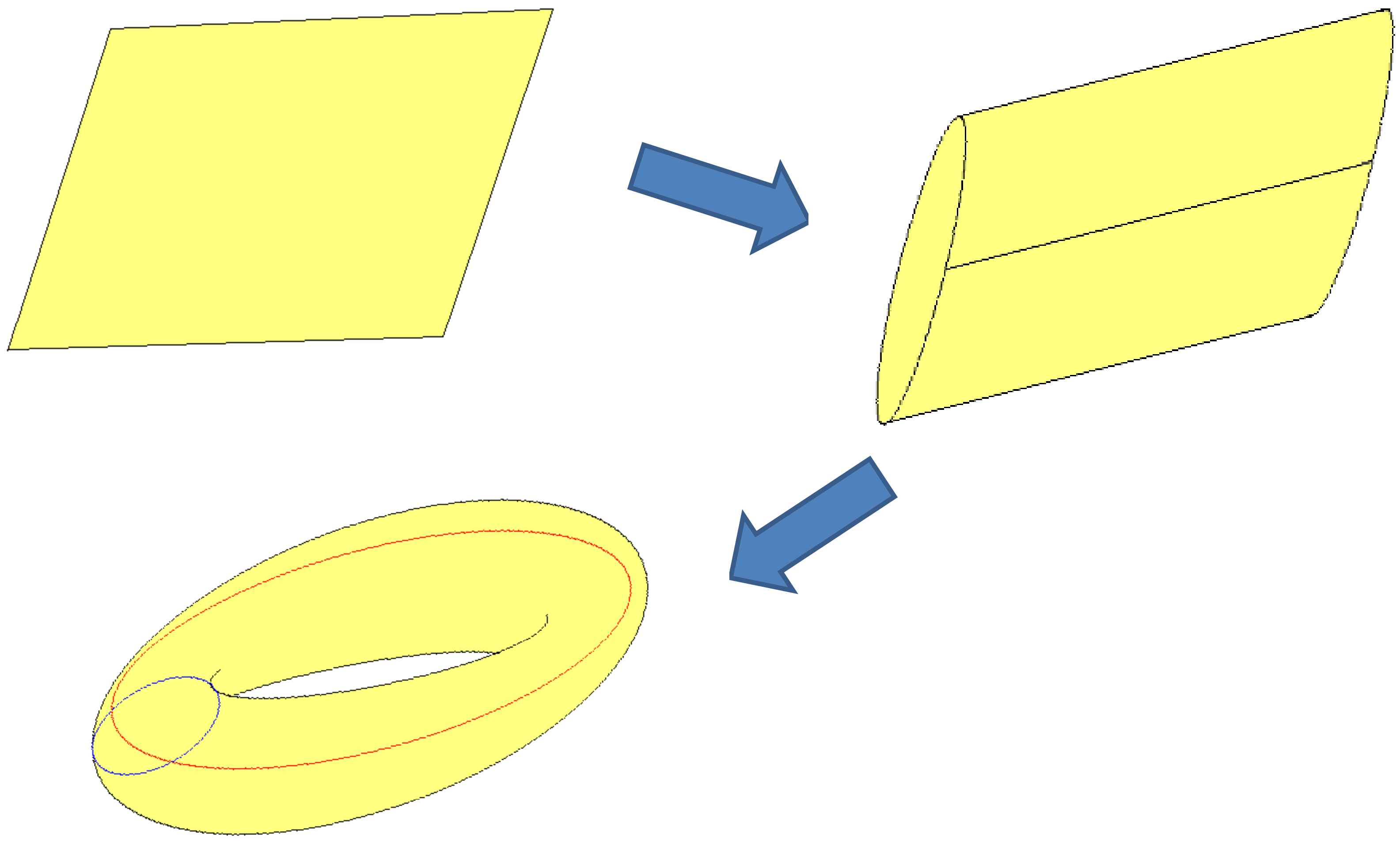
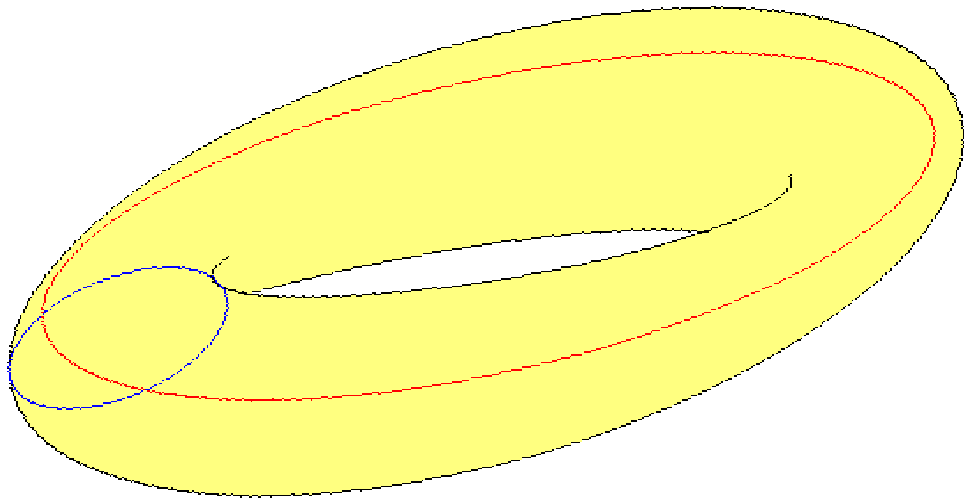


opposite sides
are identified

Euler characteristic for elliptic curves



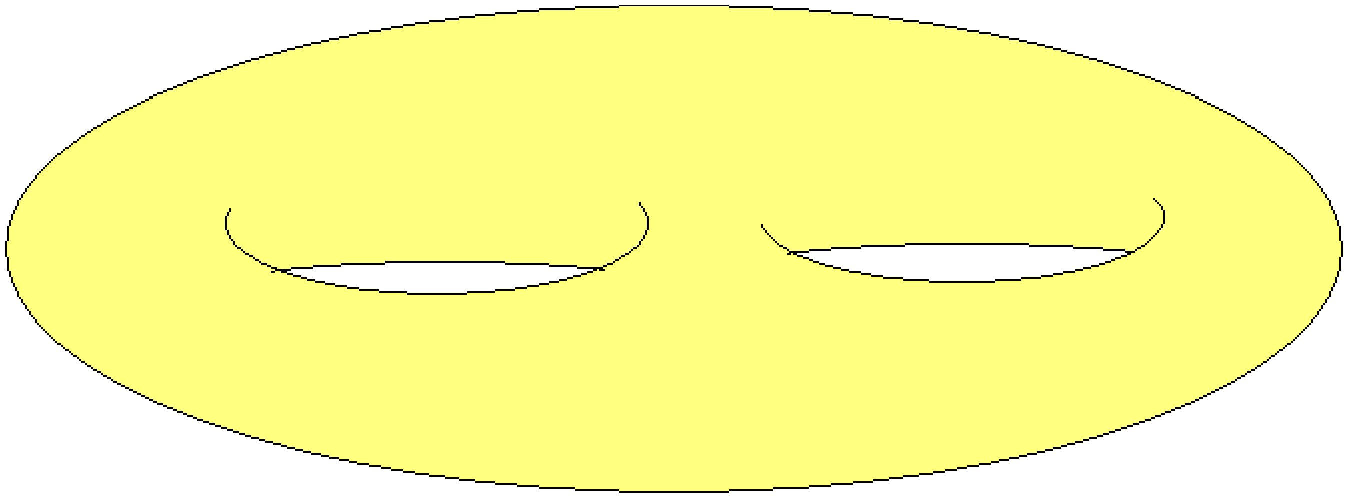
Euler characteristic for elliptic curves



$$1 \quad - \quad 2 \quad + \quad 1 \quad = \quad 0$$

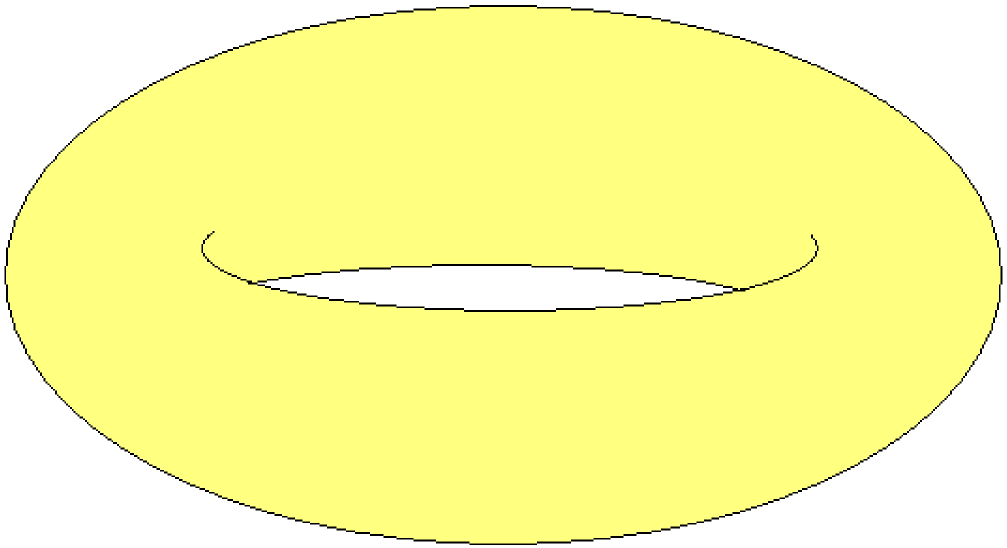
Euler characteristic for Riemann surfaces

2-2g   g : genus



g=2

Elliptic curves=Riemann surface with genus 1



Riemann surface with genus 0 =
Riemann sphere

Homology

- Riemann surface X with genus g

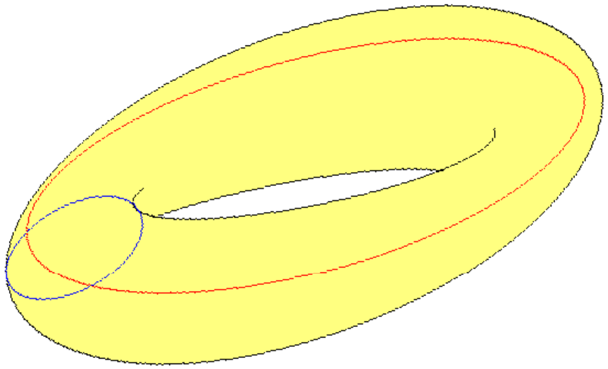
→ 2g –dimensional space $H_1(X)$

- Higher-dimensional manifolds X

→ space $H_q(X)$

- 1900 Poincaré “Analysis Situs”

Homology of elliptic curves



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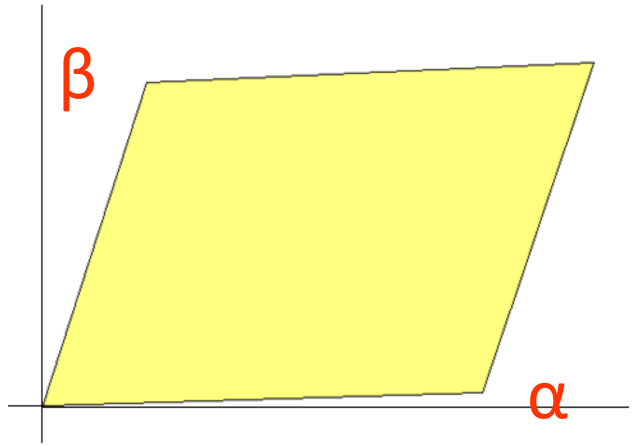

opposite sides are identified

$$H_1(E) = \{ a\alpha + b\beta \mid a, b \text{ real} \}$$

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Poincaré (1854.4.29 - 1912.7.17)

Poincaré conjecture

- “a space that has the same homology as 3-sphere”

  Is it identical to 3-sphere as a space ?

construction and detailed description of a counterexample

(Poincaré (1904))

Poincaré conjecture

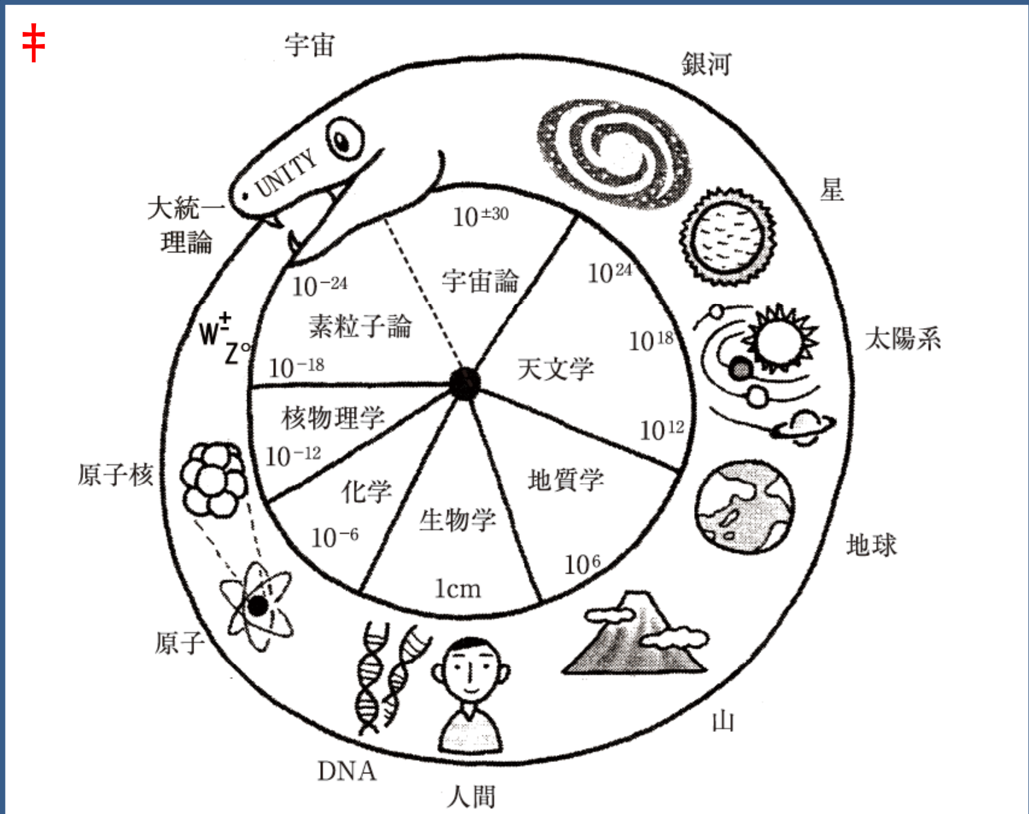
- A space has the same fundamental group as 3-sphere

  ⇒ it is identical to 3-sphere as a space

(Perelman(2003))

hierarchical structure of nature

Matter of various sizes



The size of matter physics deals with ranges from 10^{-30} cm to 10^{30}cm

ものの大きさ　By courtesy of Yasushi Suto, school of science, the university of Tokyo

Hierarchical structure in mathematics

- Individual number and figure
- Systems and spaces of numbers .... sets
- Systems of numbers and relations between spaces.... categories
- mappings that associate mathematical objects with other kinds of objects ....functors

Hierarchical structure in mathematics crude objects

numbers

0

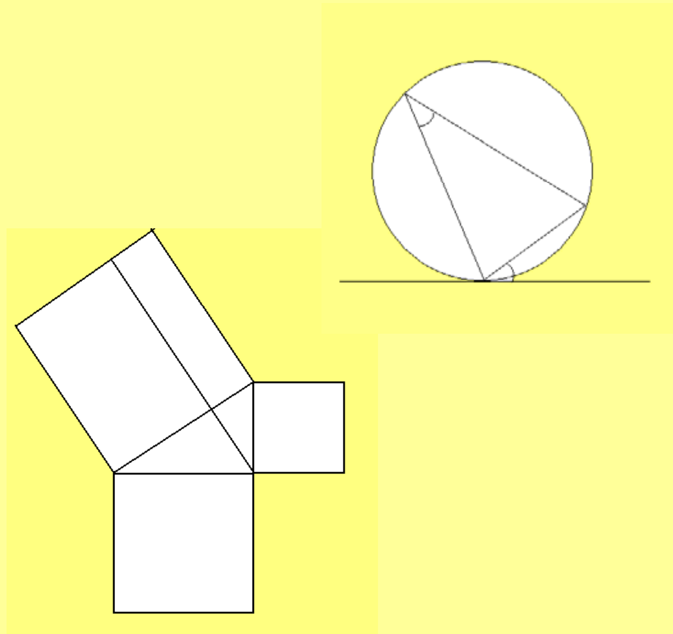
$\frac{-1 \pm \sqrt{-3}}{2}$

$\sqrt{2}$

e

π

figures



Hierarchical structure in mathematics

Sets

number systems

spaces

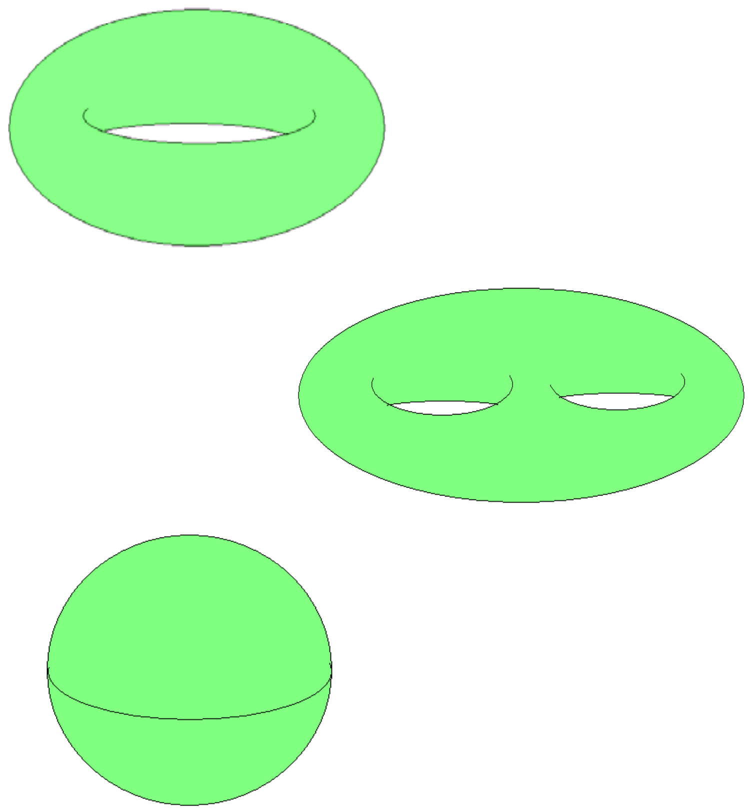
Rational number

2-adic number

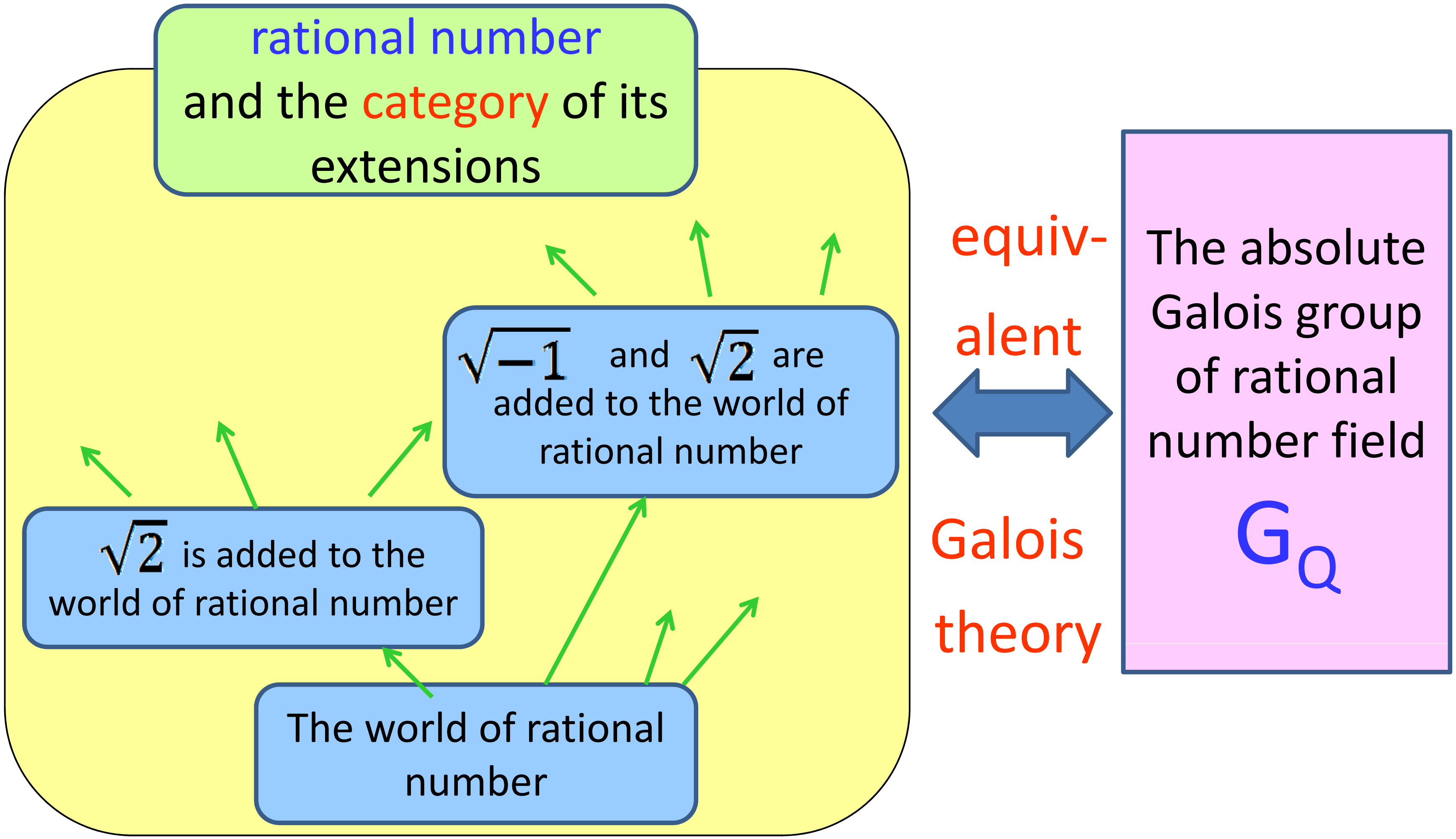
Real number

....

3-adic number



Hierarchical structure in mathematics category



Extension with equations

- $x^2 + 1 = 0$

  $a + b\sqrt{-1}$ (a, b rational number)

- $x^2 - 2 = 0$

  $a + b\sqrt{2}$ (a, b rational number)

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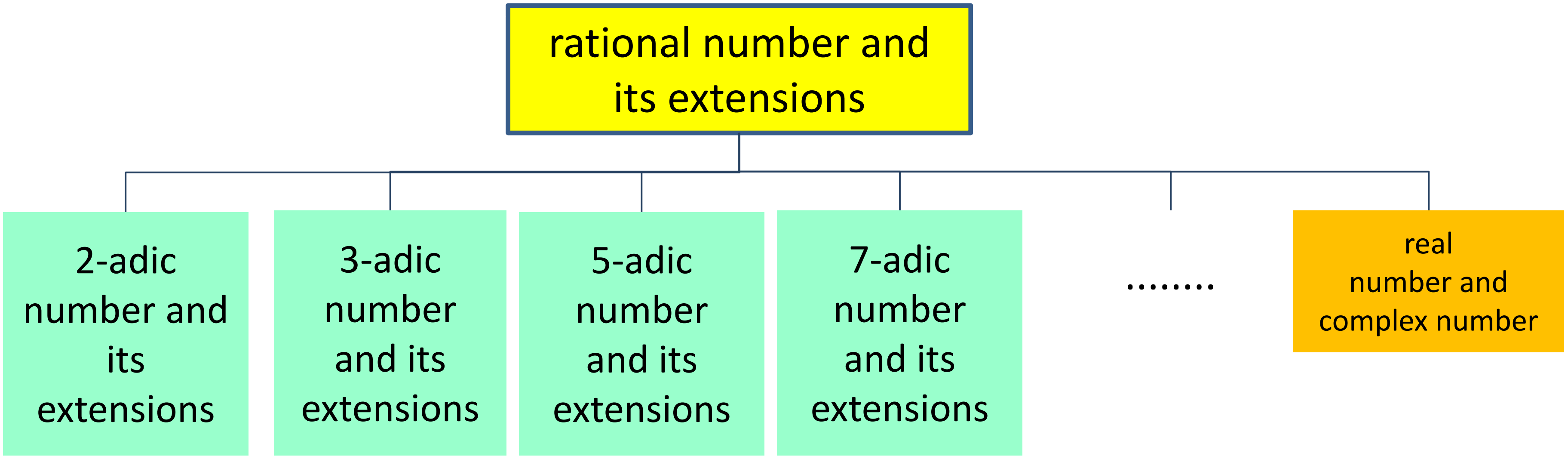
Galois (1811.10.25-1832.5.31)

The number system

(the first half of the 20th century)



- The fundamental framework of modern number theory, that enables one to reach even for the settlement of Fermat’s last theorem, was originated.

local-global principle

The absolute Galois group of rational number field G_Q

The absolute Galois group of 2-adic number field G_{Q_2}

The absolute Galois group of real number field G_R={1, complex conjugate}

The absolute Galois group of 3-adic number field G_{Q_3}

The absolute Galois group of 5-adic number field G_{Q_5}

....

local-global principle (in the case that x^n-1)

- n , $a \geq 1$: natural number, coprime

  there are infinite number of prime numbers that leave a remainder of a when divided by n :theorem on arithmetic progressions (1837)

remainder when divided by 4:

2, 3, 5, 7, 11, 13, 17, 23, 29, 31, 37, 41,

43, 47, 53, 59, 61, 67, 71, 73, 79, 83, ….

Dirichlet (1805.2.13–1859.5.5)

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local-global principle

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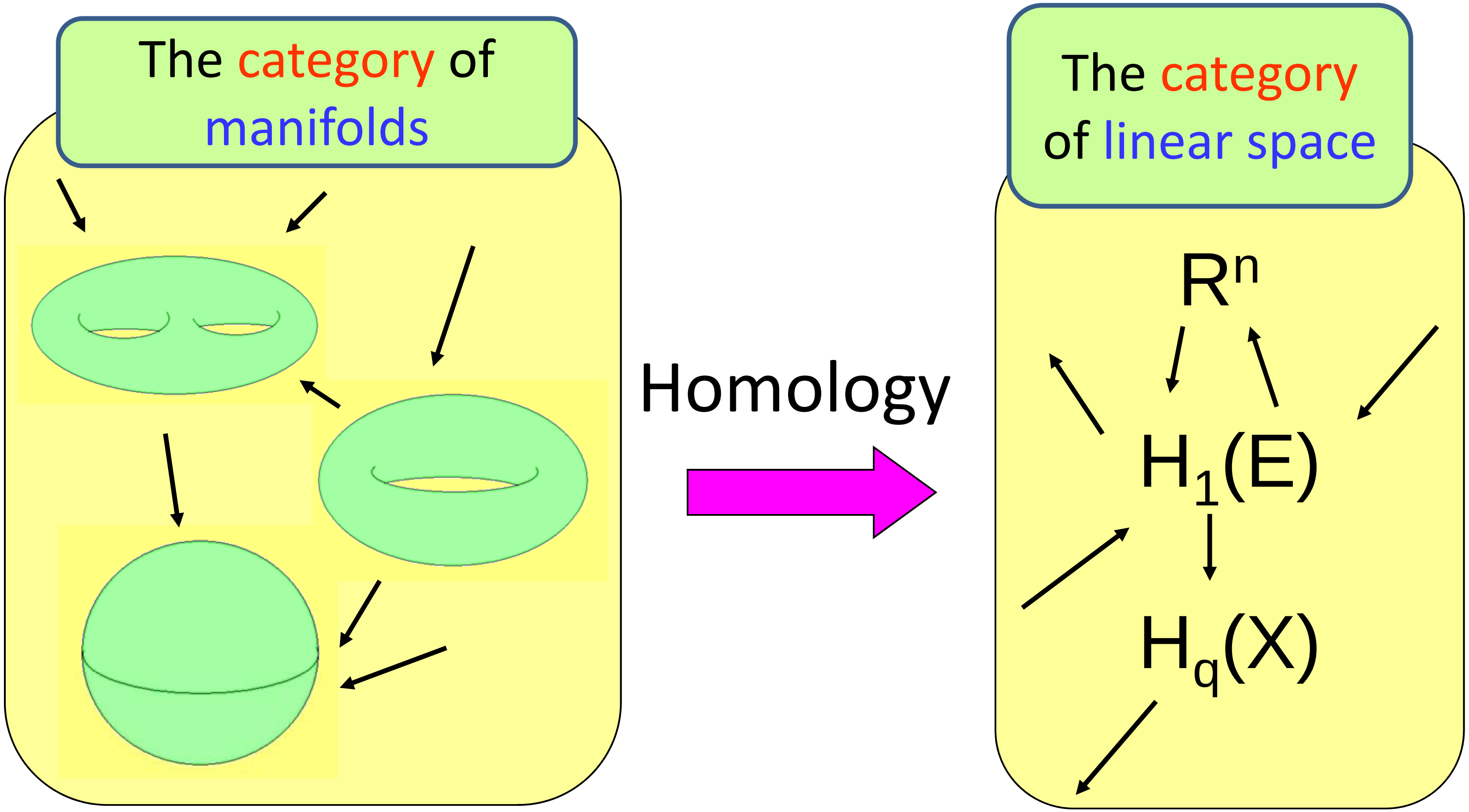
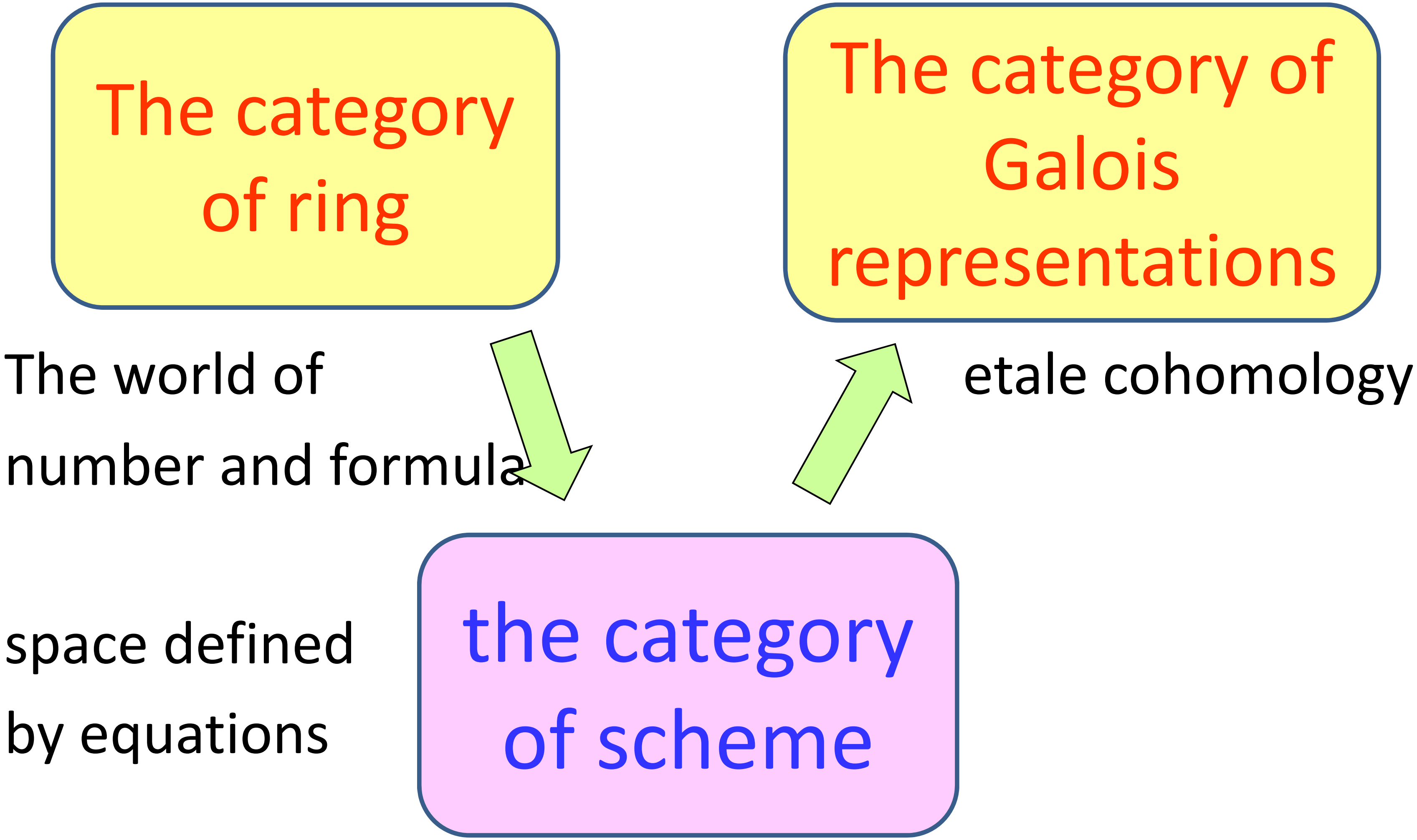
The absolute Galois group of 5-adic number field G_{Q_5}

....

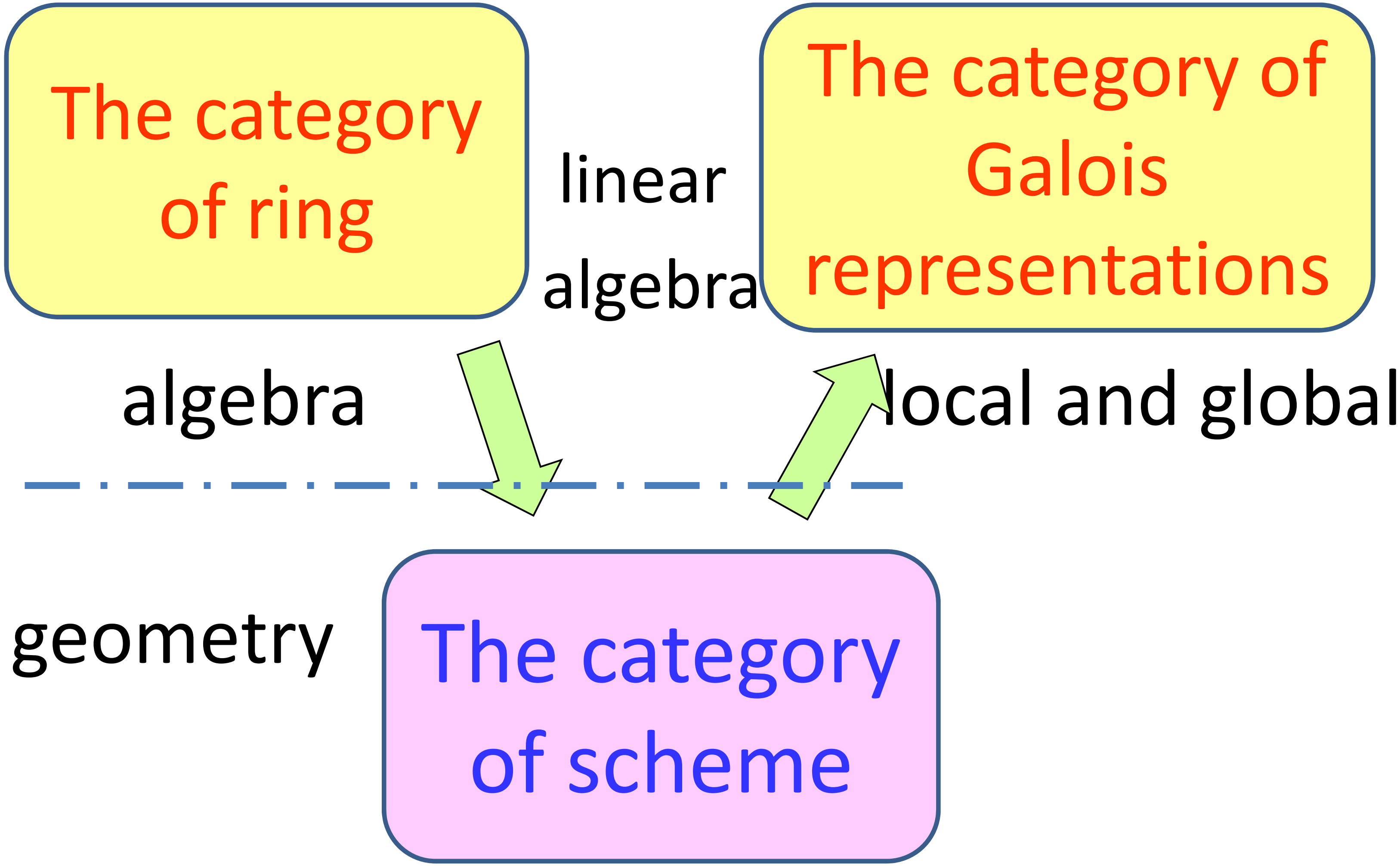
The Hierarchical structure of mathematics Functor



The world of arithmetic geometry



The world of arithmetic geometry



Elliptic curves and Galois representation

elliptic curves E

↓

Galois representation $H_1(E, Q_p)$

$= \{ a\alpha + b\beta \mid$ a, b are p-adic number $\}$

Elliptic curves and Galois representation

The "same" Galois representation $H_1(E, Q_p)$

⇩

the "same" elliptic curve E

(Faltings 1983)

establishment of arithmetic geometry

similarities between number and formula

- Division with remainders

$$32 \div 5 = 6 \text{ remainder } 2$$

$$x^5 \div (x^2 + 1) = x^3 - x \text{ remainder } x$$

similarities between number and formula

- Prime number decomposition

$$15624 = 2^3 \cdot 3^2 \cdot 7 \cdot 31$$

$$x^6 - 1 = (x - 1)(x + 1)(x^2 - x + 1)(x^2 + x + 1)$$

Similarities between number and point

$$\frac{1}{1-x}$$ a function on the number line

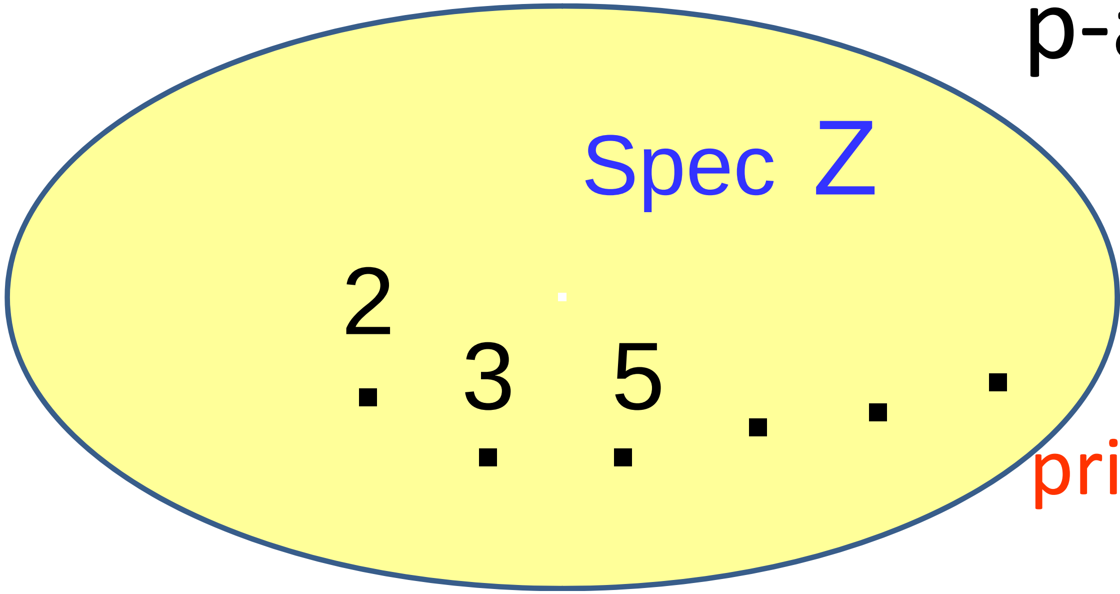
$= 1 + x + x^2 + ...$ Taylor expansion

the behavior around the point $x = 0$

Similarities between number and point

$$\frac{1}{1-p} = 1 + p + p^2 + \ldots.$$

p-adic expansion



prime number …point

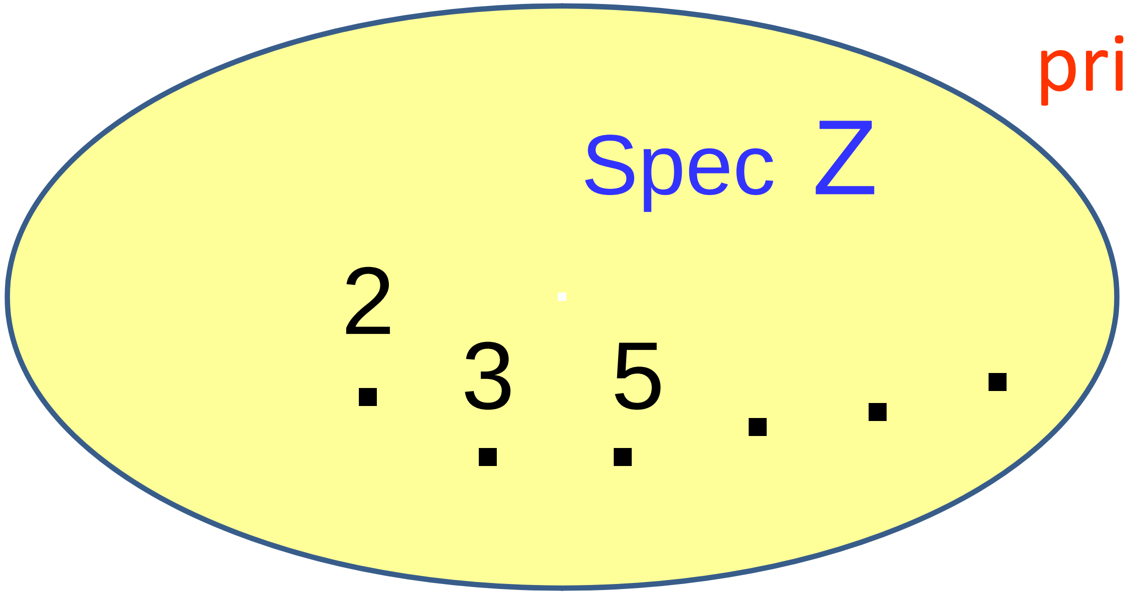
Integer … function

number and point duality

function =determines the value at a point

point = determines the value of function

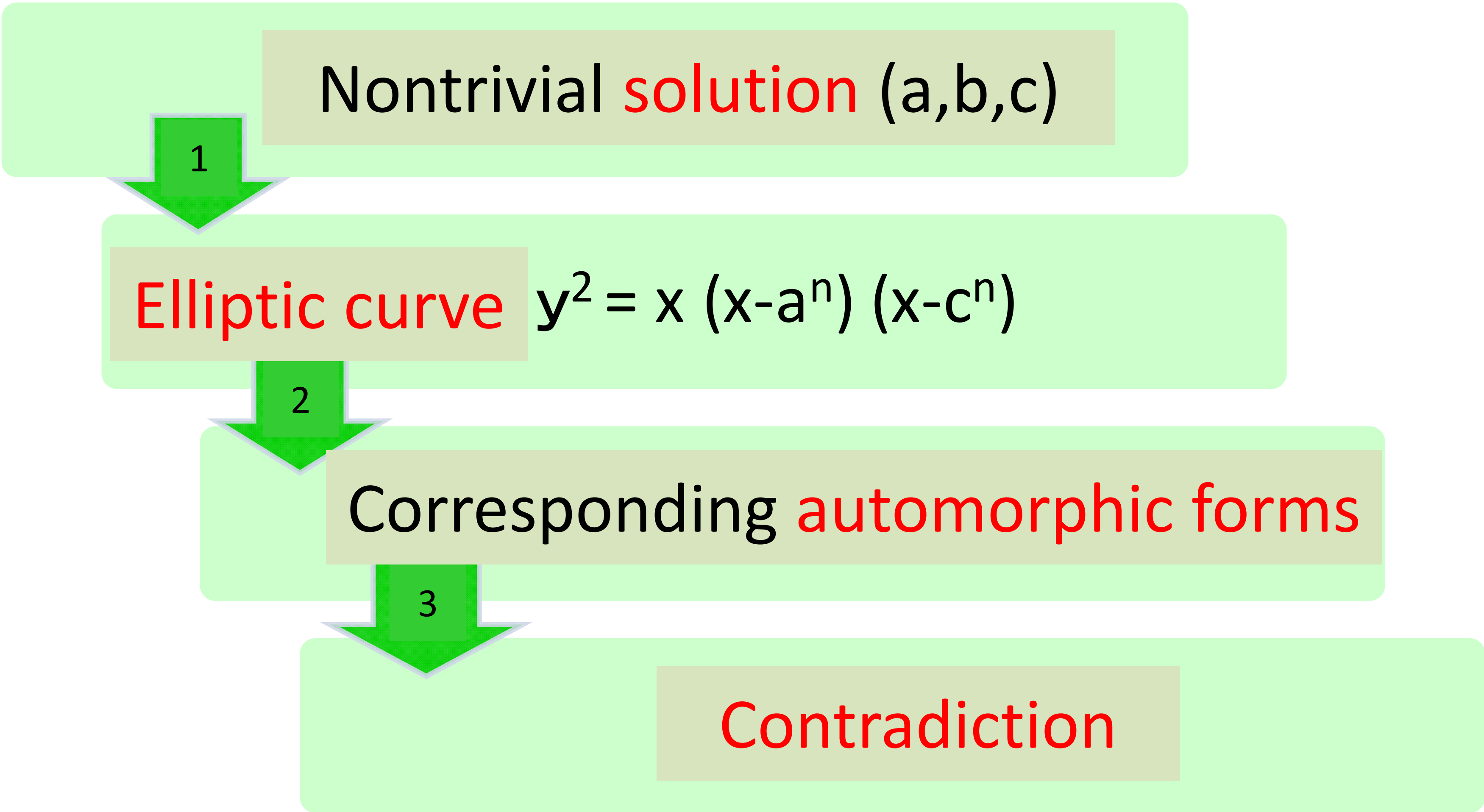
$= 1 + p + p^2 + ...$



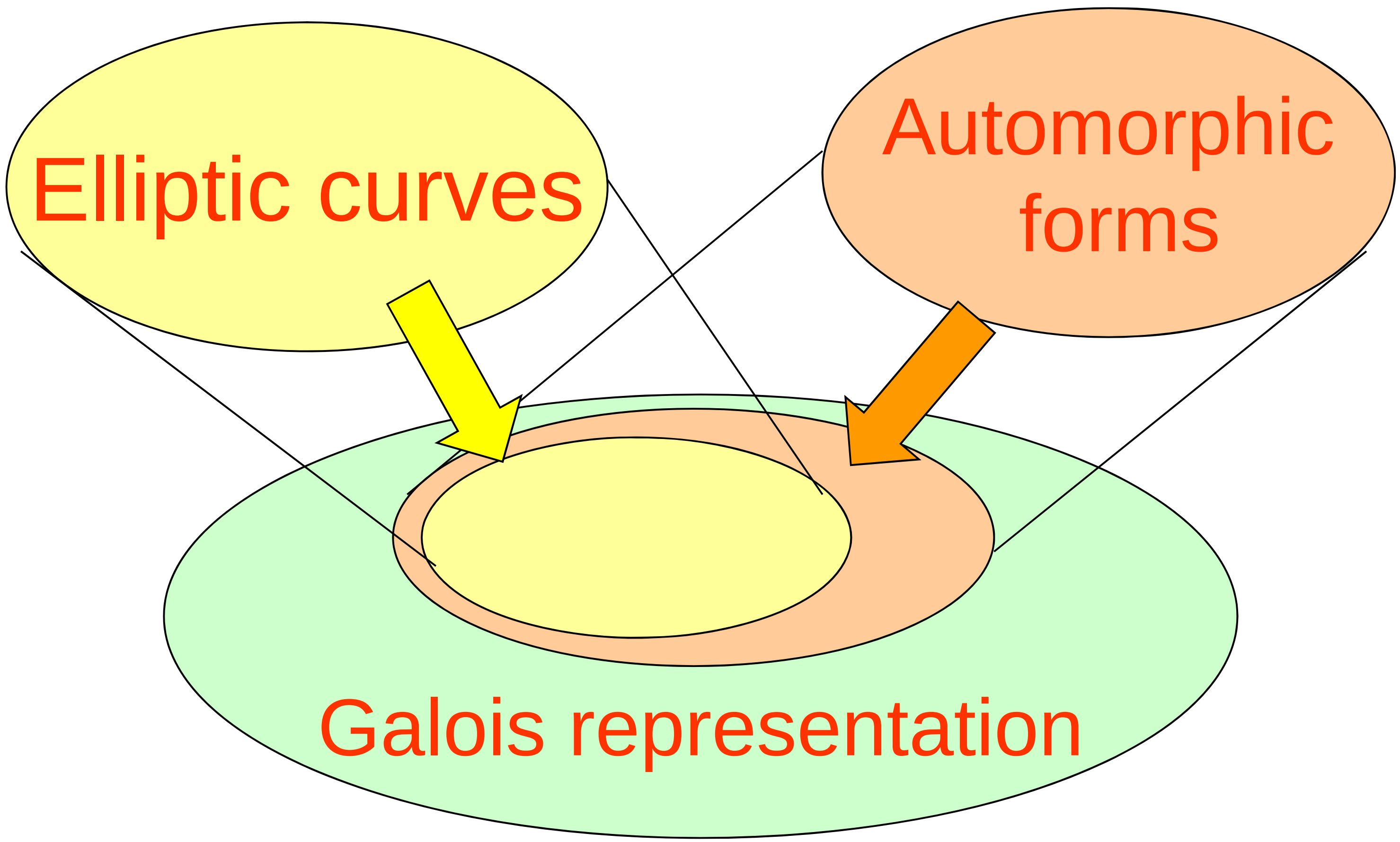
prime number …point

Integer … function

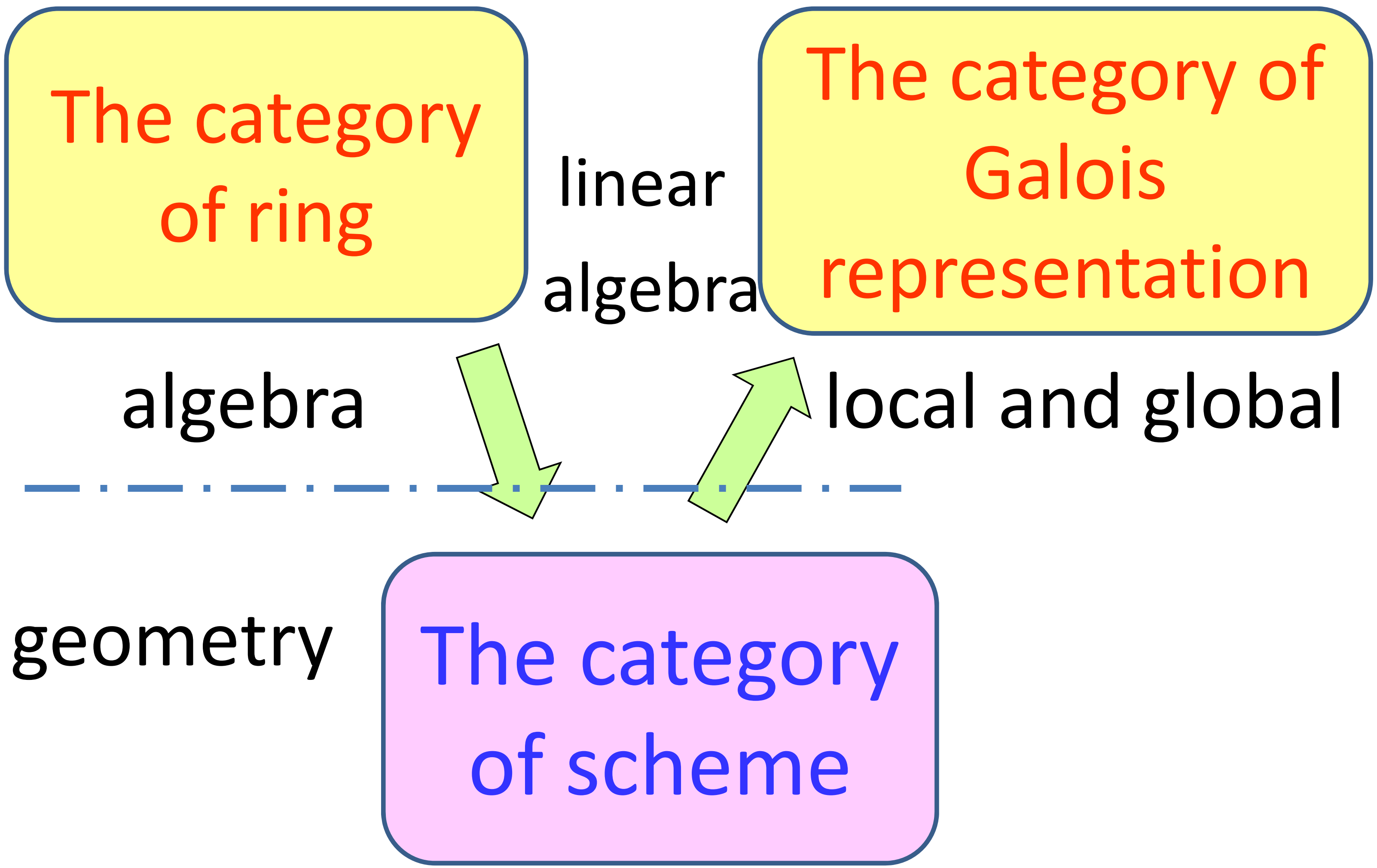
The proof of Fermat's last theorem



The settlement of Taniyama-Shimura conjecture



The world of arithmetic geometry



The settlement of Taniyama-Shimura conjecture